Parametric Relations and Inverses
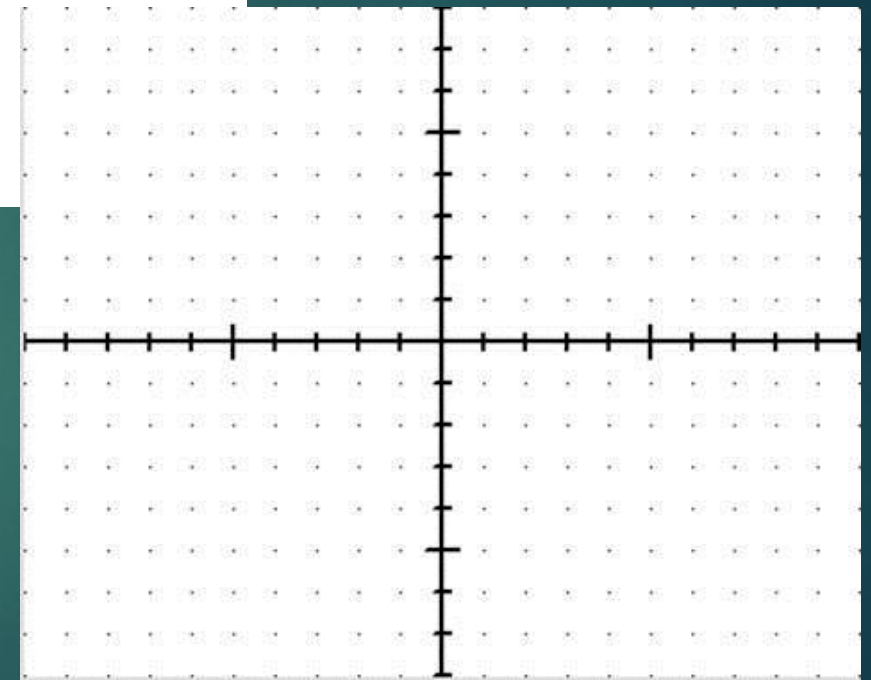
Defining a Function Parametrically

Another natural way to define functions or, more generally, relations, is to define *both* elements of the ordered pair (x, y) in terms of another variable t, called a **parameter**. We illustrate with an example.

Consider the set of all ordered pairs (x, y) defined by the equations

$$x = t + 1$$

$$y = t^2 + 2t$$

t	x	y

Using a Graphing Calculator in Parametric Mode

Consider the set of all ordered pairs (x, y) defined by the equations

$$x = t^2 + 2t$$

$$y = t + 1$$

Plot1   Plot2   Plot3
\X₁ᴛ=T²+2T
Y₁ᴛ=T+1
\X₂ᴛ=
Y₂ᴛ=
\X₃ᴛ=
Y₃ᴛ=
\X₄ᴛ=

TABLE SETUP
TblStart=−3
ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask

T	X₁ᴛ	Y₁ᴛ
−3	3	−2
−2	0	−1
−1	−1	0
0	0	1
1	3	2
2	8	3
3	15	4

$Y_{1T} = T+1$

(a)
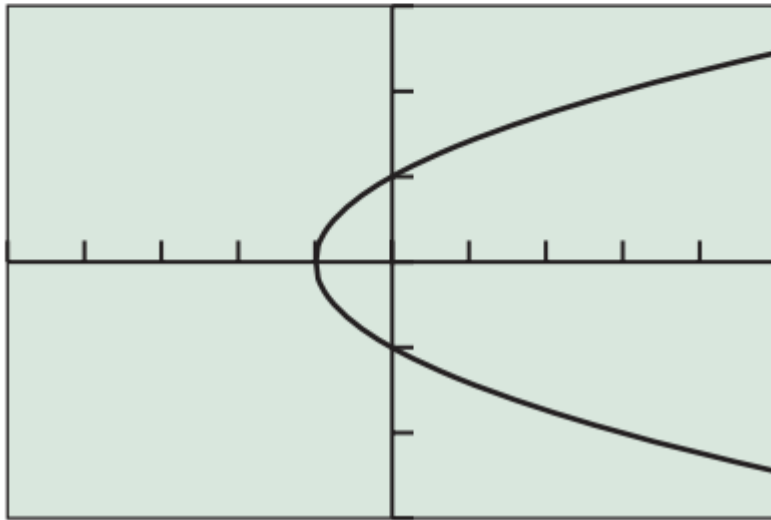
Eliminate the parameter:

DEFINITION **Inverse Relation**

The ordered pair (a, b) is in a relation if and only if the ordered pair (b, a) is in the **inverse relation**.
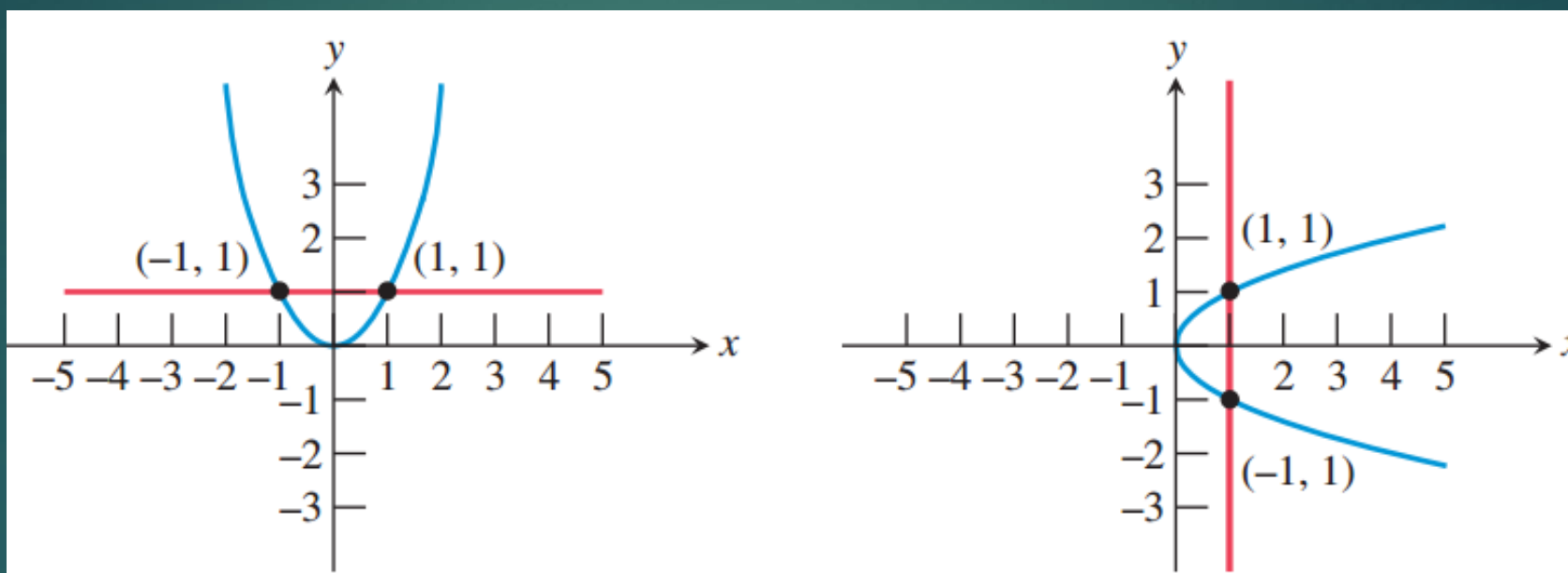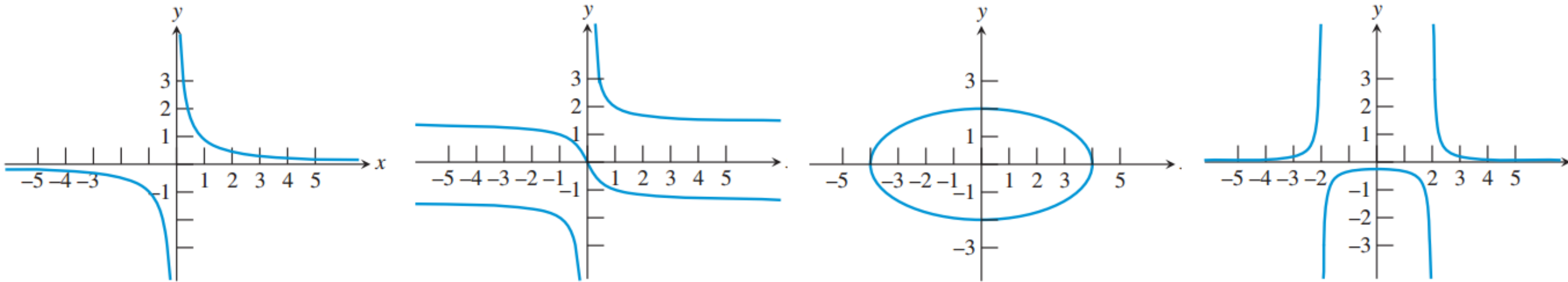

Horizontal Line Test

The inverse of a relation is a function if and only if each horizontal line intersects the graph of the original relation in at most one point.

Applying the Horizontal Line Test

Which of the graphs (1)–(4) in Figure 1.66 are graphs of

(a) relations that are functions?

(b) relations that have inverses that are functions?

A *function* whose inverse is a function has a graph that passes both the *horizontal and vertical* line tests (such as graph (1) in Example 3). Such a function is **one-to-one**, since every x is paired with a unique y and every y is paired with a unique x.

DEFINITION Inverse Function

If f is a one-to-one function with domain D and range R, then the **inverse function of f**, denoted f^{-1}, is the function with domain R and range D defined by

$$f^{-1}(b) = a \quad \text{if and only if} \quad f(a) = b.$$
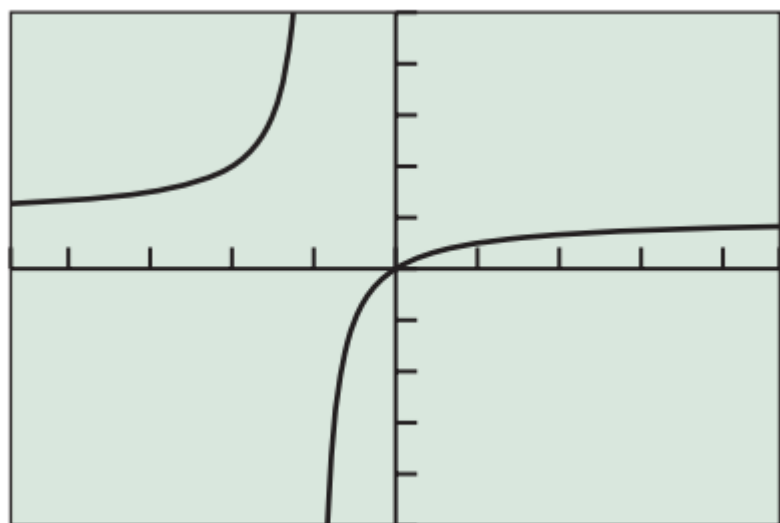
Finding an Inverse Function Algebraically

Find an equation for $f^{-1}(x)$ if $f(x) = x/(x + 1)$.

SOLUTION The graph of f in Figure 1.67 suggests that f is one-to-one. The original function satisfies the equation $y = x/(x + 1)$. If f truly is one-to-one, the inverse function f^{-1} will satisfy the equation $x = y/(y + 1)$. (Note that we just switch the x and the y.)

If we solve this new equation for y we will have a formula for $f^{-1}(x)$:

$$x = \frac{y}{y + 1}$$