Average Velocity

Average velocity is the change in position divided by the change in time, as in the following familiar-looking example.

EXAMPLE 1 Computing Average Velocity

An automobile travels 120 miles in 2 hours and 30 minutes. What is its average velocity over the entire $2\frac{1}{2}$ hour time interval?

EXAMPLE 2 Using Limits to Avoid Zero Division

A ball rolls down a ramp so that its distance s from the top of the ramp after t seconds is exactly t^2 feet. What is its instantaneous velocity after 3 seconds?

On the interval [3, 3.1]:

$$\frac{\Delta s}{\Delta t} = \frac{(3.1)^2 - 3^2}{3.1 - 3} = \frac{0.61}{0.1} = 6.1 \text{ feet per second.}$$

On the interval [3, 3.05]:

$$\frac{\Delta s}{\Delta t} = \frac{(3.05)^2 - 3^2}{3.05 - 3} = \frac{0.3025}{0.05} = 6.05 \text{ feet per second.}$$

However, we can see *directly* what is happening to the quotient by treating it as a *limit* of the average velocity on the interval $[3, t]$ as t approaches 3:

$$\begin{aligned}
\lim_{t\to 3} \frac{\Delta s}{\Delta t} &= \lim_{t\to 3} \frac{t^2 - 3^2}{t - 3} \\
&= \lim_{t\to 3} \frac{(t+3)(t-3)}{t-3} && \text{Factor the numerator.} \\
&= \lim_{t\to 3} (t+3) \\
&= 6
\end{aligned}$$

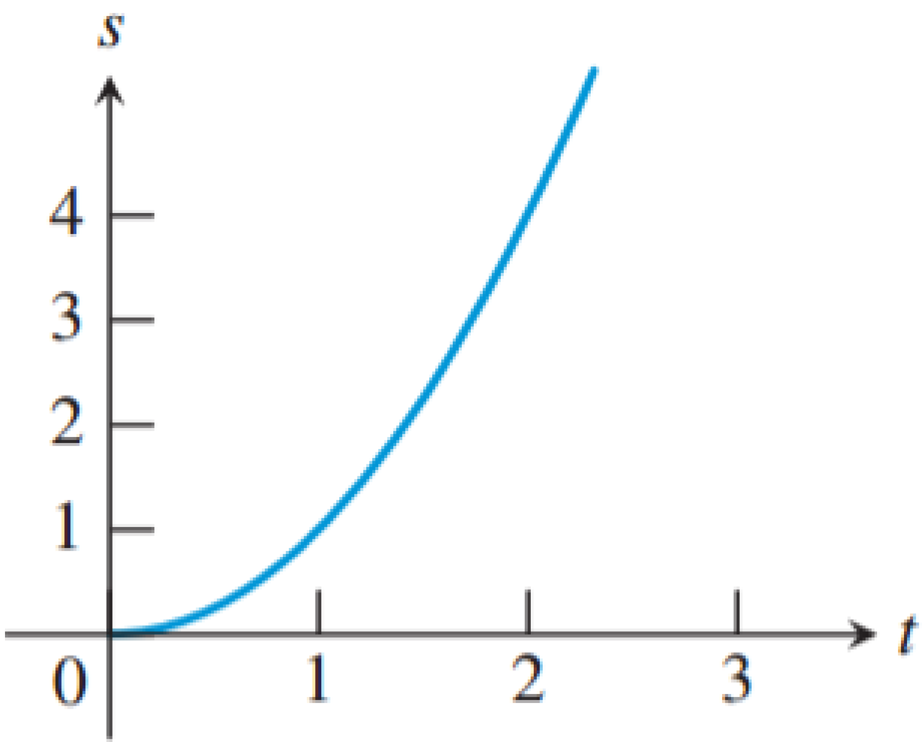
DEFINITION (INFORMAL) Limit at *a*

When we write "$\lim_{x \to a} f(x) = L$," we mean that $f(x)$ gets arbitrarily close to L as x gets arbitrarily close (but not equal) to a.

The Connection to Tangent Lines

What Galileo discovered by rolling balls down ramps was that the distance traveled was proportional to the *square* of the elapsed time. For simplicity, let us suppose that the ramp was tilted just enough so that the relation between s, the distance from the top of the ramp, and t, the elapsed time, was given (as in Example 2) by

$$s = t^2.$$



EXPLORATION 1 Seeing Average Velocity

Copy Figure 10.1 on a piece of paper and connect the points (1, 1) and (2, 4) with a straight line. (This is called a *secant line* because it connects two points on the curve.)

1. Find the slope of the line.
2. Find the average velocity of the ball over the time interval [1, 2].
3. What is the relationship between the numbers that answer questions 1 and 2?
4. In general, how could you represent the average velocity of the ball over the time interval $[a, b]$ geometrically?

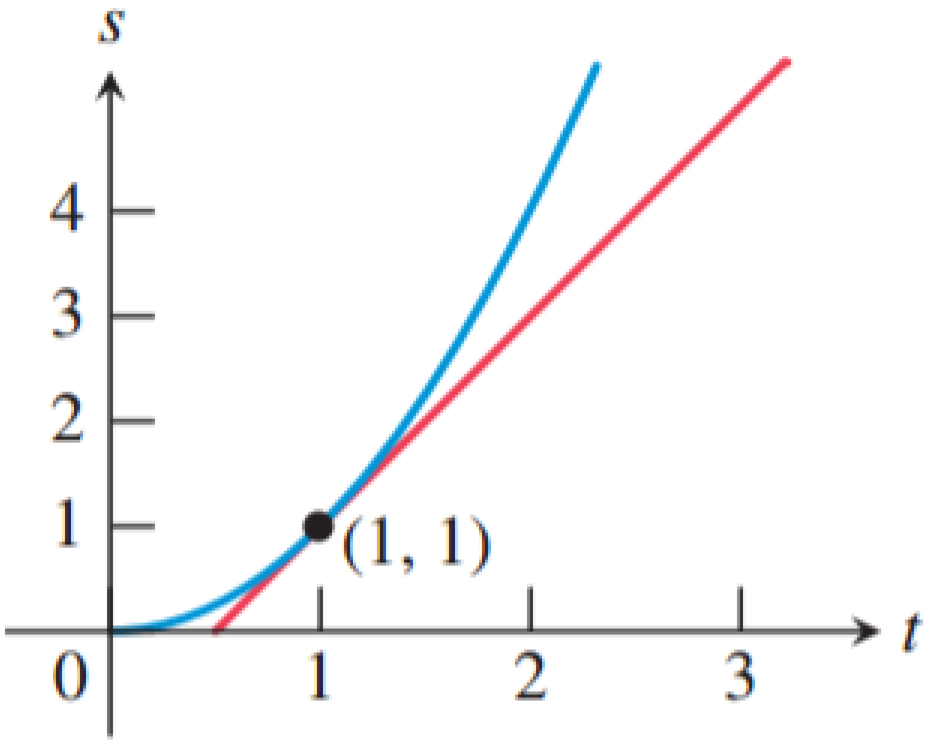


FIGURE 10.2 A line tangent to the graph of $s = t^2$ at the point (1, 1). The slope of this line appears to be the instantaneous velocity at $t = 1$, even though $\Delta s/\Delta t = 0/0$. The geometry succeeds where the algebra fails!

EXAMPLE 3 Finding the Slope of a Tangent Line

Use limits to find the slope of the tangent line to the graph of $s = t^2$ at the point $(1, 1)$ (Figure 10.2).

SOLUTION This will look a lot like the solution to Example 2.

$$\lim_{t \to 1} \frac{\Delta s}{\Delta t} = \lim_{t \to 1} \frac{t^2 - 1^2}{t - 1}$$

The Derivative

Velocity, the rate of change of position with respect to time, is only one application of the general concept of "rate of change." If $y = f(x)$ is *any* function, we can speak of how y changes as x changes.

DEFINITION Average Rate of Change

If $y = f(x)$, then the **average rate of change** of y with respect to x on the interval $[a, b]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

Geometrically, this is the slope of the **secant line** through $(a, f(a))$ and $(b, f(b))$.

DEFINITION **Derivative at a Point**

The **derivative of the function f at $x = a$**, denoted by $f'(a)$ and read "f prime of a" is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

provided the limit exists.

Geometrically, this is the slope of the **tangent line** through $(a, f(a))$.

A more computationally useful formula for the derivative is obtained by letting $x = a + h$ and looking at the limit as h approaches 0 (equivalent to letting x approach a).

DEFINITION Derivative at a Point (easier for computing)

The **derivative of the function f at $x = a$**, denoted by $f'(a)$ and read "f prime of a" is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h},$$

provided the limit exists.

EXAMPLE 4 Finding a Derivative at a Point

Find $f'(4)$ if $f(x) = 2x^2 - 3$.

SOLUTION

$$f'(4) = \lim_{h \to 0} \frac{f(4 + h) - f(4)}{h}$$

DEFINITION **Derivative**

If $y = f(x)$, then the **derivative of the function f with respect to x**, is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

for all values of x where the limit exists.

EXAMPLE 5 Finding the Derivative of a Function

(a) Find $f'(x)$ if $f(x) = x^2$.

To emphasize the connection with slope $\triangle y/\triangle x$, Leibniz used the notation dy/dx for the derivative. (The dy and dx were his "ghosts of departed quantities.") This **Leibniz notation** has several advantages over the "prime" notation, as you will learn when you study calculus. We will use both notations in our examples and exercises.

(b) Find $\dfrac{dy}{dx}$ if $y = \dfrac{1}{x}$.