Limits and Motion: The Area Problem

EXAMPLE 1 Computing Distance Traveled

An automobile travels at a constant rate of 48 miles per hour for 2 hours and 30 minutes. How far does the automobile travel?

SOLUTION We apply the formula $d = rt$:

$$d = (48 \text{ mi/hr})(2.5 \text{ hr}) = 120 \text{ miles.}$$

Distance from a Changing Velocity

A Distance Question

Suppose a ball rolls down a ramp and its velocity is always $2t$ feet per second, where t is the number of seconds after it started to roll. How far does the ball travel during the first 3 seconds?

Velocity times Δt gives Δs. But instantaneous velocity occurs at an instant of time, so $\Delta t = 0$. That means $\Delta s = 0$. So, at any given instant of time, the ball doesn't move. Since any time interval consists of instants of time, the ball never moves at all! (You might well ask: Is this another trick question?)

As was the case with the Velocity Question in Section 10.1, this foolish-looking example conceals a very subtle algebraic dilemma—and, far from being a trick question, it is exactly the question that needed to be answered in order to compute the distance traveled by an object whose velocity varies as a function of time. The scientists who were working on the tangent line problem realized that the distance-traveled problem must be related to it, but, surprisingly, their geometry led them in another direction. The distance traveled problem led them not to tangent lines, but to areas.

Limits at Infinity

Before we see the connection to areas, let us revisit another limit concept that will make instantaneous velocity easier to handle, just as in the last section. We will again be content with an informal definition.

DEFINITION (INFORMAL) Limit at Infinity

When we write "$\lim_{x \to \infty} f(x) = L$," we mean that $f(x)$ gets arbitrarily close to L as x gets arbitrarily large.

EXPLORATION 1 **An Infinite Limit**

A gallon of water is divided equally and poured into teacups. Find the amount in each teacup and the *total amount* in *all* the teacups if there are

1. 10 teacups
2. 100 teacups
3. 1 billion teacups
4. an infinite number of teacups

The preceding Exploration probably went pretty smoothly until you came to the infinite number of teacups. At that point you were probably pretty comfortable in saying what the *total amount* would be, and probably a little uncomfortable in saying how much would be in each teacup. (Theoretically it would be zero, which is just one reason why the actual experiment cannot be performed.) In the language of limits, the total amount of water in the infinite number of teacups would look like this:

$$\lim_{n\to\infty}\left(n\cdot\frac{1}{n}\right)=\lim_{n\to\infty}\frac{n}{n}=1\text{ gallon}$$

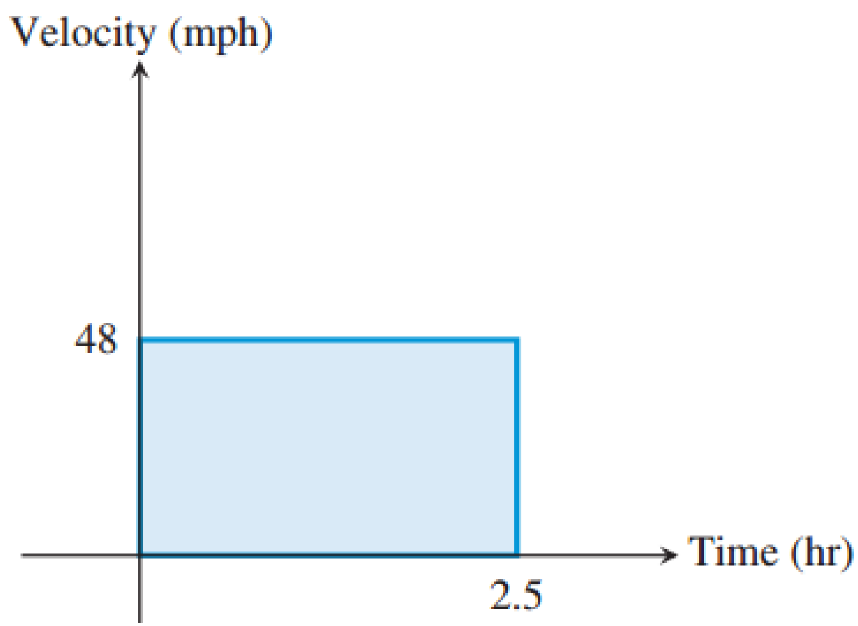
while the total amount in each teacup would look like this:

$$\lim_{n\to\infty}\frac{1}{n}=0\text{ gallons.}$$

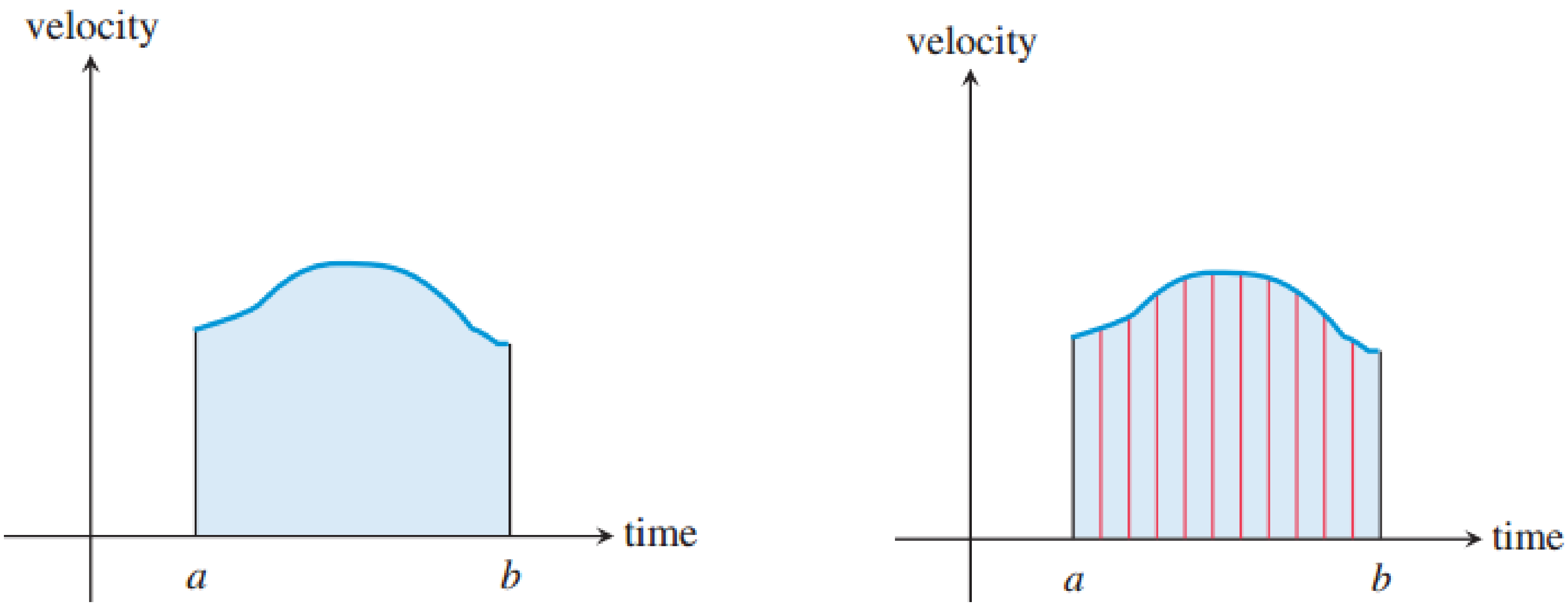
The Connection to Areas

If we graph the constant velocity $v = 48$ in Example 1 as a function of time t, we notice that the area of the shaded rectangle is the same as the distance traveled (Figure 10.4). This is no mere coincidence, either, as the area of the rectangle and the distance traveled over the time interval are both computed by multiplying the same two quantities:

$$(48 \text{ mph})(2.5 \text{ hr}) = 120 \text{ miles}.$$

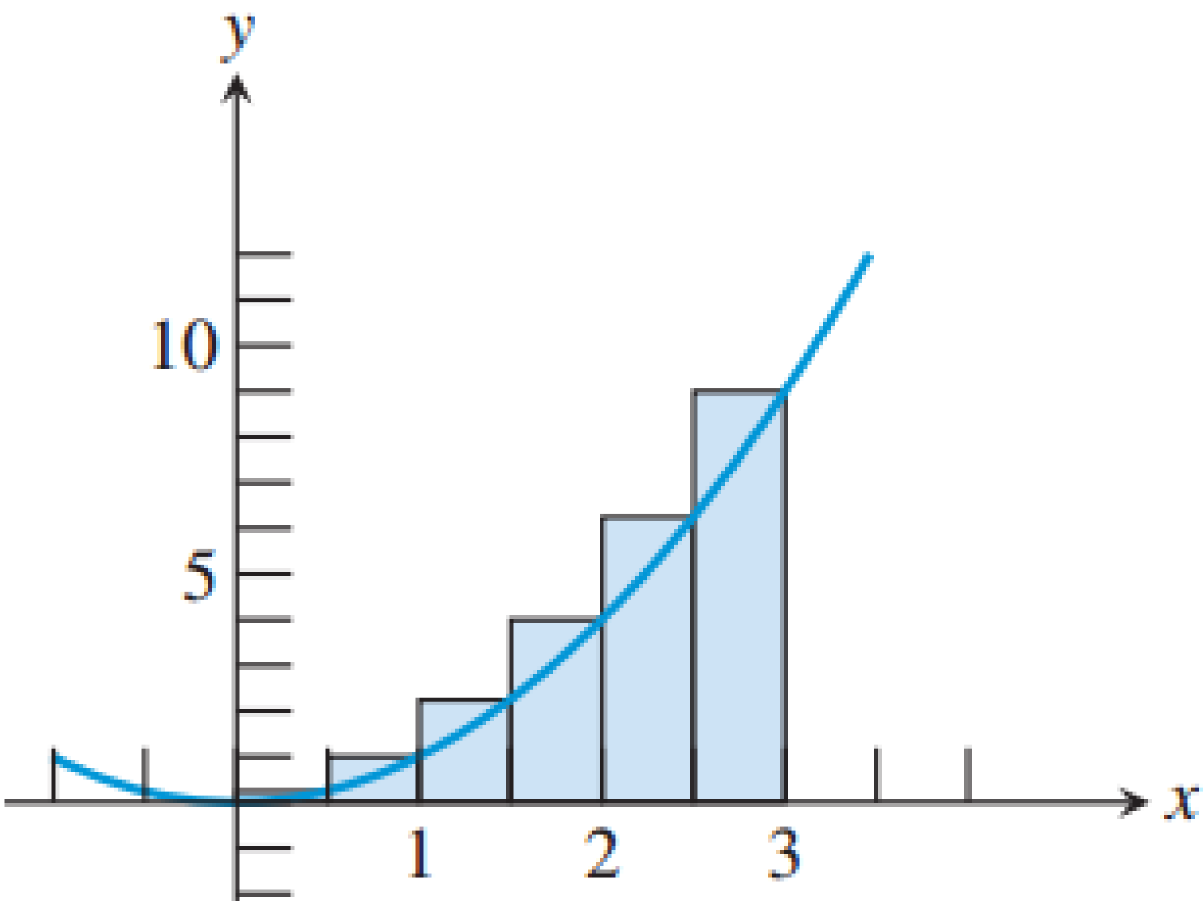


Now suppose we graph a velocity function that varies continuously as a function of time (Figure 10.5). Would the area of this irregularly-shaped region still give the total distance traveled over the time interval $[a, b]$?



EXAMPLE 3 Approximating an Area with Rectangles

Use the six rectangles in Figure 10.7 to approximate the area of the region below the graph of $f(x) = x^2$ over the interval $[0, 3]$.



SOLUTION The base of each approximating rectangle is 1/2. The height is determined by the function value at the right-hand endpoint of each subinterval. The areas of the six rectangles and the total area are computed in the table below:

Subinterval	Base of rectangle	Height of rectangle	Area of rectangle
[0, 1/2]	1/2	$f(1/2) = (1/2)^2 = 1/4$	$(1/2)(1/4) = 0.125$
[1/2, 1]	1/2	$f(1) = (1)^2 = 1$	$(1/2)(1) = 0.500$
[1, 3/2]	1/2	$f(3/2) = (3/2)^2 = 9/4$	$(1/2)(9/4) = 1.125$
[3/2, 2]	1/2	$f(2) = (2)^2 = 4$	$(1/2)(4) = 2.000$
[2, 5/2]	1/2	$f(5/2) = (5/2)^2 = 25/4$	$(1/2)(25/4) = 3.125$
[5/2, 3]	1/2	$f(3) = (3)^2 = 9$	$(1/2)(9) = 4.500$
		Total Area:	11.375

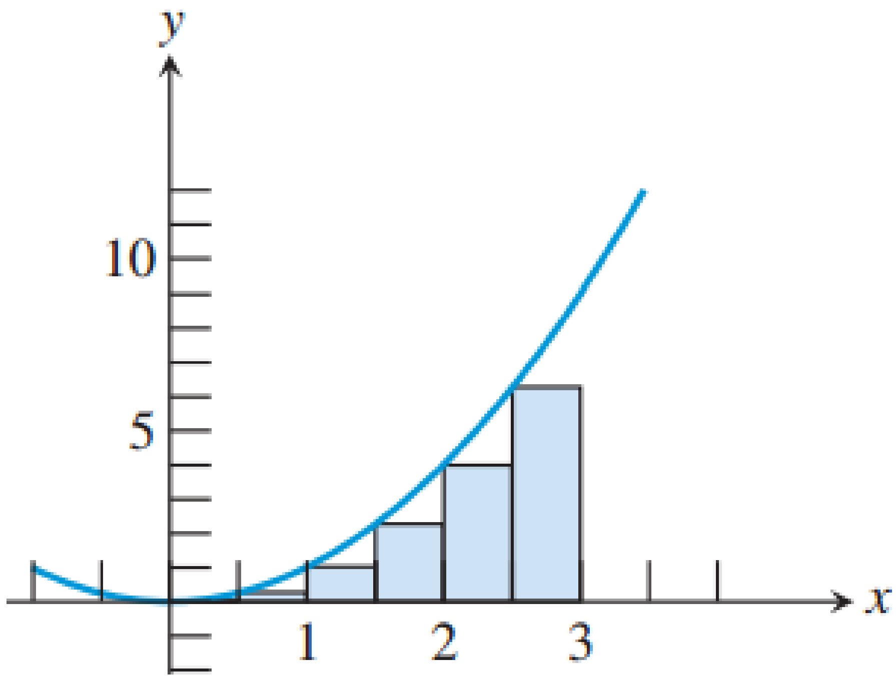


Figure 10.7 shows that the *right rectangular approximation method* (RRAM) in Example 4 overestimates the true area. If we were to use the function values at the left-hand end-points of the subintervals (LRAM), we would obtain a rectangular approximation (6.875 square units) that underestimates the true area (Figure 10.8). The average of the two approximations is 9.125 square units, which is actually a pretty good estimate of the true area of 9 square units. If we were to repeat the process with 20 rectangles, the average would be 9.01125.

The calculus step is to move from a finite number of rectangles (yielding an approximate area) to an infinite number of rectangles (yielding an exact area). This brings us to the definite integral.

The Definite Integral

In general, begin with a continuous function $y = f(x)$ over an interval $[a, b]$. Divide $[a, b]$ into n subintervals of length $\Delta x = (b - a)/n$. Choose any value x_1 in the first subinterval, x_2 in the second, and so on. Compute $f(x_1), f(x_2), f(x_3), \ldots, f(x_n)$, multiply each value by Δx, and sum up the products. In sigma notation, the sum of the products is

$$\sum_{i=1}^{n} f(x_i)\Delta x.$$

The *limit* of this sum as n approaches infinity is the solution to the area problem, and hence the solution to the problem of distance traveled. Indeed, it solves a variety of other problems as well, as you will learn when you study calculus. The limit, if it exists, is called a *definite integral*.

DEFINITION Definite Integral

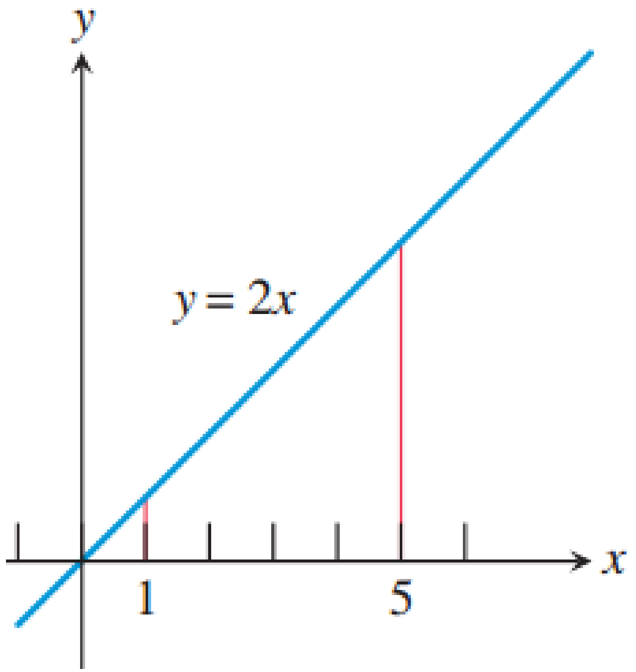
Let f be a function defined on $[a, b]$ and let $\sum_{i=1}^{n} f(x_i)\Delta x$ be defined as above. The definite integral of f over $[a, b]$, denoted $\int_a^b f(x)\,dx$, is given by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

provided the limit exists. If the limit exists, we say f is **integrable** on $[a, b]$.

EXAMPLE 4 Computing an Integral

Find $\int_1^5 2x\,dx$.



EXAMPLE 5 Computing an Integral

Suppose a ball rolls down a ramp so that its velocity after t seconds is always $2t$ feet per second. How far does it fall during the first 3 seconds?