Defining a Limit Informally

There is nothing difficult about the following limit statements:

$$\lim_{x \to 3} (2x - 1) = 5 \qquad \lim_{x \to \infty} (x^2 + 3) = \infty \qquad \lim_{n \to \infty} \frac{1}{n} = 0$$

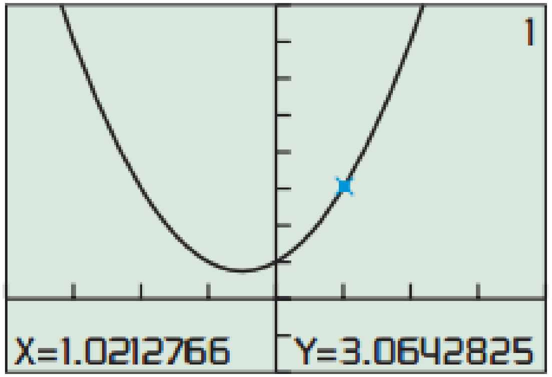
DEFINITION (INFORMAL) Limit at a

When we write "$\lim_{x \to a} f(x) = L$," we mean that $f(x)$ gets arbitrarily close to L as x gets arbitrarily close (but not equal) to a.

EXAMPLE 1 Finding a Limit

Find $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

Solve Graphically



[−4, 4] by [−2, 8]

Solve Numerically

X	Y1	
.997	2.991	
.998	2.994	
.999	2.997	
1	ERROR	
1.001	3.003	
1.002	3.006	
1.003	3.009	

Y1 = (X3–1)/(X–1)

(b)

Solve Algebraically

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1}$$

Properties of Limits

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then

1. Sum Rule	$\lim_{x\to c} (f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
2. Difference Rule	$\lim_{x\to c} (f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$
3. Product Rule	$\lim_{x\to c} (f(x) \cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
4. Constant Multiple Rule	$\lim_{x\to c} (k \cdot g(x)) = k \cdot \lim_{x\to c} g(x)$
5. Quotient Rule	$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$
6. Power Rule	$\lim_{x\to c} (f(x))^n = (\lim_{x\to c} f(x))^n$ for n a positive integer
7. Root Rule	$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ for $n \geq 2$ a positive integer, provided $\sqrt[n]{\lim_{x\to c} f(x)}$ and $\lim_{x\to c} \sqrt[n]{f(x)}$ are real numbers.

EXAMPLE 2 Using the Limit Properties

You will learn in Example 10 that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. Use this fact, along with the limit properties, to find the following limits:

(a) $\lim_{x \to 0} \frac{x + \sin x}{x}$ **(b)** $\lim_{x \to 0} \frac{1 - \cos^2 x}{x^2}$ **(c)** $\lim_{x \to 0} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{x}}$

Limits of Continuous Functions

Recall from Section 1.2 that a function is continuous at a if $\lim_{x \to a} f(x) = f(a)$. This means that the limit (at a) of a function can be found by "plugging in a" provided the function is continuous at a. (The condition of continuity is essential when employing this strategy. For example, plugging in 0 does not work on any of the limits in Example 2.)

EXAMPLE 3 Finding Limits by Substitution

Find the limits.

(a) $\displaystyle \lim_{x \to 0} \frac{e^x - \tan x}{\cos^2 x}$

(b) $\displaystyle \lim_{n \to 16} \frac{\sqrt{n}}{\log_2 n}$

One-sided and Two-sided Limits

We can see that the limit of the function in Figure 10.11 is 3 whether x approaches 1 from the left or right. Sometimes the values of a function f can approach different values as x approaches a number c from opposite sides. When this happens, the limit of f as x approaches c from the left is the **left-hand limit** of f at c and the limit of f as x approaches c from the right is the **right-hand limit** of f at c. Here is the notation we use:

left-hand: $\lim_{x \to c^-} f(x)$ *The limit of f as x approaches c from the left.*

right-hand: $\lim_{x \to c^+} f(x)$ *The limit of f as x approaches c from the right.*

EXAMPLE 4 Finding Left- and Right-Hand Limits

Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$ where $f(x) = \begin{cases} -x^2 + 4x - 1 & \text{if } x \le 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$.

THEOREM One-sided and Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the left-hand and right-hand limits at c exist and are equal. That is,

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L.$$

EXAMPLE 5 Finding a Limit at a Point of Discontinuity

Let

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3. \end{cases}$$

Find $\lim_{x \to 3} f(x)$ and prove that f is discontinuous at $x = 3$.

Limits Involving Infinity

DEFINITION **Limits at Infinity**

When we write "$\lim_{x \to \infty} f(x) = L$," we mean that $f(x)$ gets arbitrarily close to L as x gets arbitrarily large. We say that ***f* has a limit *L* as *x* approaches ∞**.

When we write "$\lim_{x \to -\infty} f(x) = L$," we mean that $f(x)$ gets arbitrarily close to L as $-x$ gets arbitrarily large. We say that ***f* has a limit *L* as *x* approaches $-\infty$**.

Notice that limits, whether at a or at infinity, are always finite real numbers; otherwise, the limits do not exist. For example, it is correct to write

$$\lim_{x\to 0} \frac{1}{x^2} \text{ does not exist,}$$

since it approaches no real number L. In this case, however, it is also convenient to write

$$\lim_{x\to 0} \frac{1}{x^2} = \infty,$$

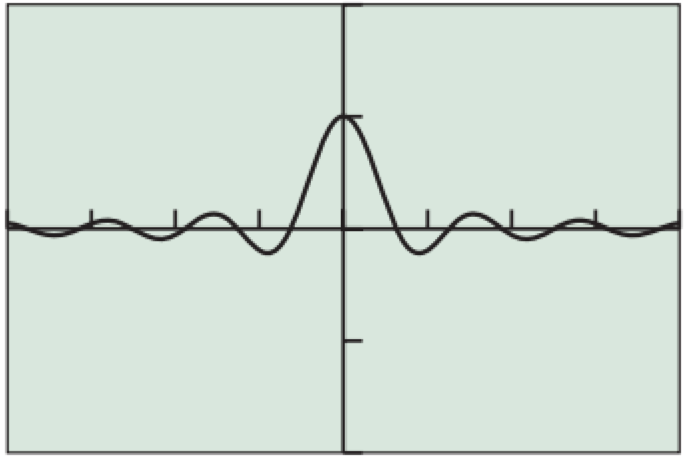
which gives us a little more information about *why* the limit fails to exist. (It increases without bound.) Similarly, it is convenient to write

$$\lim_{x\to 0^+} \ln x = -\infty,$$

since $\ln x$ decreases without bound as x approaches 0 from the right. In this context, the symbols "∞" and "$-\infty$" are sometimes called **infinite limits**.

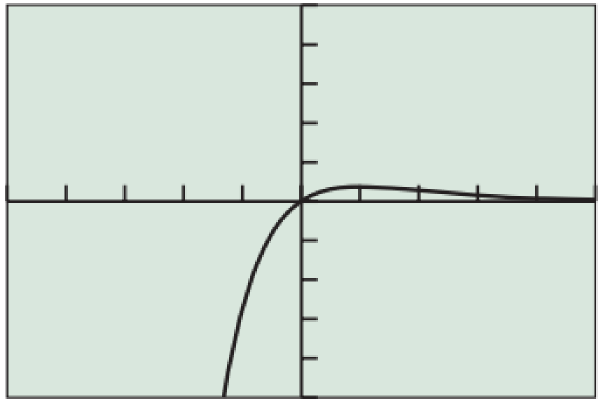
EXAMPLE 7 Investigating Limits as $x \to \pm\infty$

Let $f(x) = (\sin x)/x$. Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.



EXAMPLE 8 Using Tables to Investigate Limits as $x \to \pm\infty$

Let $f(x) = xe^{-x}$. Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.



[–5, 5] by [–5, 5]

X	Y2	
0	0	
10	4.5E–4	
20	4.1E–8	
30	3E–12	
40	2E–16	
50	1E–20	
60	5E–25	

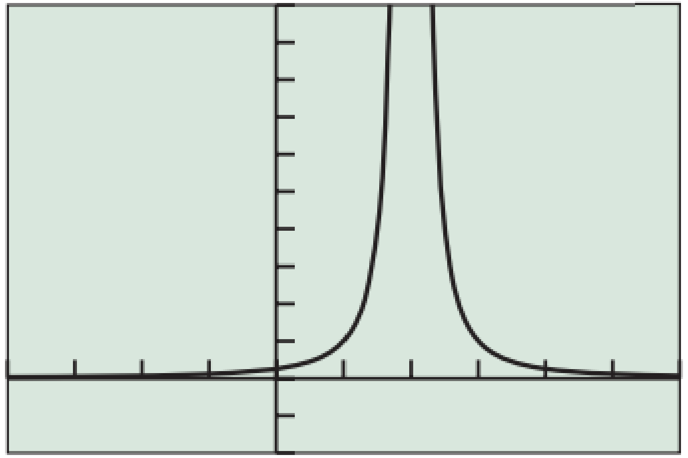
Y2 = Xe^(–X)

X	Y2	
0	0	
–10	–2.2E5	
–20	–9.7E9	
–30	–3E14	
–40	–9E18	
–50	–3E23	
–60	–7E27	

Y2 = Xe^(–X)

EXAMPLE 9 Investigating Unbounded Limits

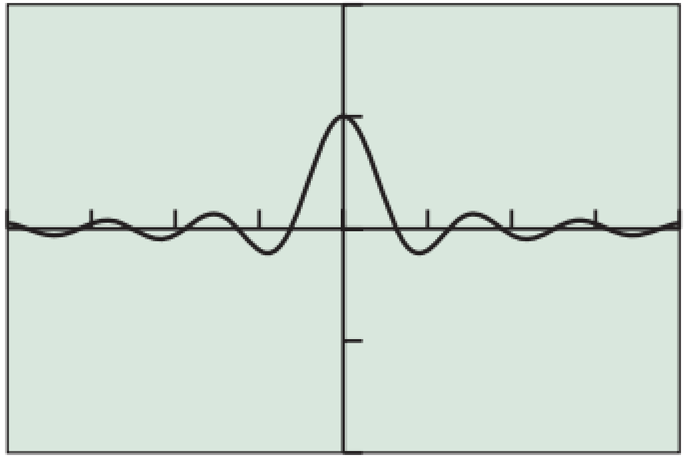
Find $\lim_{x\to 2} 1/(x-2)^2$.



[–4, 6] by [–2, 10]

EXAMPLE 10 Investigating a Limit at $x = 0$

Find $\lim_{x \to 0} (\sin x)/x$.



X	Y_1
−.03	.99985
−.02	.99993
−.01	.99998
0	ERROR
.01	.99998
.02	.99993
.03	.99985

$Y_1 = \sin(X)/X$