DEFINITION Polynomial Function

Let n be a nonnegative integer and let $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is a **polynomial function of degree n**. The **leading coefficient** is a_n.

The zero function $f(x) = 0$ is a polynomial function. It has no degree and no leading coefficient.

EXAMPLE 1 Identifying Polynomial Functions

Which of the following are polynomial functions? For those that are polynomial functions, state the degree and leading coefficient. For those that are not, explain why not.

(a) $f(x) = 4x^3 - 5x - \frac{1}{2}$

(b) $g(x) = 6x^{-4} + 7$

(c) $h(x) = \sqrt{9x^4 + 16x^2}$

(d) $k(x) = 15x - 2x^4$

Polynomial Functions of No and Low Degree

Name	Form	Degree
Zero function	$f(x) = 0$	Undefined
Constant function	$f(x) = a \ (a \neq 0)$	0
Linear function	$f(x) = ax + b \ (a \neq 0)$	1
Quadratic function	$f(x) = ax^2 + bx + c \ (a \neq 0)$	2

Linear Functions and Their Graphs

A **linear function** is a polynomial function of degree 1 and so has the form

$$f(x) = ax + b, \text{ where } a \text{ and } b \text{ are constants and } a \neq 0.$$

Vertical lines are not graphs of functions because they fail the vertical line test, and horizontal lines are graphs of constant functions. A line in the Cartesian plane is the graph of a linear function if and only if it is a **slant line**, that is, neither horizontal nor vertical.

EXAMPLE 2 Finding an Equation of a Linear Function

Write an equation for the linear function f such that $f(-1) = 2$ and $f(3) = -2$.

SOLUTION

We seek a line through the points $(-1, 2)$ and $(3, -2)$. The slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-2) - 2}{3 - (-1)} = \frac{-4}{4} = -1.$$

$$y - 2 = -1(x - (-1))$$

Average Rate of Change

Another property that characterizes a linear function is its *rate of change*. The **average rate of change** of a function $y = f(x)$ between $x = a$ and $x = b$, $a \neq b$, is

$$\frac{f(b) - f(a)}{b - a}.$$

THEOREM Constant Rate of Change

A function defined on all real numbers is a linear function if and only if it has a constant nonzero average rate of change between any two points on its graph.

Because the average rate of change of a linear function is constant, it is called simply the **rate of change** of the linear function. The slope m in the formula $f(x) = mx + b$ is the rate of change of the linear function.

EXPLORATION 1 Modeling Depreciation with a Linear Function

Camelot Apartments bought a \$50,000 building and for tax purposes are depreciating it \$2000 per year over a 25-yr period using straight-line depreciation.

1. What is the rate of change of the value of the building?
2. Write an equation for the value $v(t)$ of the building as a linear function of the time t since the building was placed in service.
3. Evaluate $v(0)$ and $v(16)$.
4. Solve $v(t) = 39{,}000$.

$$\text{rate of change} = m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}.$$

Vertex Form of a Quadratic Function

Any quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, can be written in the **vertex form**

$$f(x) = a(x - h)^2 + k.$$

The graph of f is a parabola with vertex (h, k) and axis $x = h$, where $h = -b/(2a)$ and $k = c - ah^2$. If $a > 0$, the parabola opens upward, and if $a < 0$, it opens downward. (See Figure 2.6.)

Finding the Vertex and Axis of a Quadratic Function

Use the vertex form of a quadratic function to find the vertex and axis of the graph of $f(x) = 6x - 3x^2 - 5$. Rewrite the equation in vertex form.

Using Algebra to Describe the Graph of a Quadratic Function

Use completing the square to describe the graph of $f(x) = 3x^2 + 12x + 11$.