Long Division and the Division Algorithm

112	$1x^2 + 1x + 2$	← Quotient
$32\overline{)3587}$	$3x + 2\overline{)3x^3 + 5x^2 + 8x + 7}$	← Dividend
$\underline{32}$	$\underline{3x^3 + 2x^2}$	← Multiply: $1x^2 \cdot (3x + 2)$
387	$3x^2 + 8x + 7$	← Subtract
$\underline{32}$	$\underline{3x^2 + 2x}$	← Multiply: $1x \cdot (3x + 2)$
67	$6x + 7$	← Subtract
$\underline{64}$	$\underline{6x + 4}$	← Multiply: $2 \cdot (3x + 2)$
3	3	← Remainder

(Divisor)(Quotient) + Remainder = Dividend.

$$32 \times 112 + 3 = 3587$$

$$(3x + 2)(x^2 + x + 2) + 3 = 3x^3 + 5x^2 + 8x + 7.$$

Division Algorithm for Polynomials

Let $f(x)$ and $d(x)$ be polynomials with the degree of f greater than or equal to the degree of d, and $d(x) \neq 0$. Then there are unique polynomials $q(x)$ and $r(x)$, called the **quotient** and **remainder**, such that

$$f(x) = d(x) \cdot q(x) + r(x)$$

where either $r(x) = 0$ or the degree of r is less than the degree of d.

EXAMPLE 1 Using Polynomial Long Division

Use long division to find the quotient and remainder when $2x^4 - x^3 - 2$ is divided by $2x^2 + x + 1$. Write a summary statement in both polynomial and fraction form.

$$2x^2 + x + 1 \overline{\smash{)}2x^4 - x^3 + 0x^2 + 0x - 2}$$

Remainder and Factor Theorems

THEOREM Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

we evaluate the polynomial $f(x)$ at $x = k$:

EXAMPLE 2 Using the Remainder Theorem

Find the remainder when $f(x) = 3x^2 + 7x - 20$ is divided by

(a) $x - 2$ **(b)** $x + 1$ **(c)** $x + 4$.

THEOREM Factor Theorem

A polynomial function $f(x)$ has a factor $x - k$ if and only if $f(k) = 0$.

Applying the ideas of the Factor Theorem to Example 2, we can factor $f(x) = 3x^2 + 7x - 20$ by dividing it by the known factor $x + 4$.

$$
\begin{array}{r}
3x - 5 \\
x + 4 \overline{)\, 3x^2 + 7x - 20} \\
\underline{3x^2 + 12x } \\
-5x - 20 \\
\underline{-5x - 20} \\
0
\end{array}
$$

So, $f(x) = 3x^2 + 7x - 20 = (x + 4)(3x - 5)$.

Fundamental Connections for Polynomial Functions

For a polynomial function f and a real number k, the following statements are equivalent:

1. $x = k$ is a solution (or root) of the equation $f(x) = 0$.

2. k is a zero of the function f.

3. k is an x-intercept of the graph of $y = f(x)$.

4. $x - k$ is a factor of $f(x)$.

Synthetic Division

Long Division

$$
\begin{array}{r}
2x^2 + 3x \quad + 4 \\
x - 3\overline{)2x^3 - 3x^2 - 5x - 12} \\
\underline{2x^3 - 6x^2 \qquad\qquad} \\
3x^2 - 5x - 12 \\
\underline{3x^2 - 9x \qquad} \\
4x - 12 \\
\underline{4x - 12} \\
0
\end{array}
$$

$$
\begin{array}{llrrrrl}
\text{Zero of divisor} \rightarrow & 3 \,| & 2 & -3 & -5 & -12 & \text{Dividend} \\
 & & & 6 & 9 & 12 & \\
\hline
 & & 2 & 3 & 4 & 0 & \text{Quotient, remainder}
\end{array}
$$

EXAMPLE 3 Using Synthetic Division

Divide $2x^3 - 3x^2 - 5x - 12$ by $x - 3$ using synthetic division and write a summary statement in fraction form.

$$\frac{2x^3 - 3x^2 - 5x - 12}{x - 3} = 2x^2 + 3x + 4, x \neq 3.$$

Rational Zeros Theorem

Real zeros of polynomial functions are either **rational zeros**—zeros that are rational numbers—or **irrational zeros**—zeros that are irrational numbers. For example,

$$f(x) = 4x^2 - 9 = (2x + 3)(2x - 3)$$

has the rational zeros $-3/2$ and $3/2$, and

$$f(x) = x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$$

has the irrational zeros $-\sqrt{2}$ and $\sqrt{2}$.

THEOREM Rational Zeros Theorem

EXAMPLE 5 Finding the Rational Zeros

Find the rational zeros of $f(x) = 3x^3 + 4x^2 - 5x - 2$.

Potential Rational Zeros:

$$\frac{\text{Factors of } -2}{\text{Factors of } 3}: \frac{\pm 1, \pm 2}{\pm 1, \pm 3}: \pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}$$