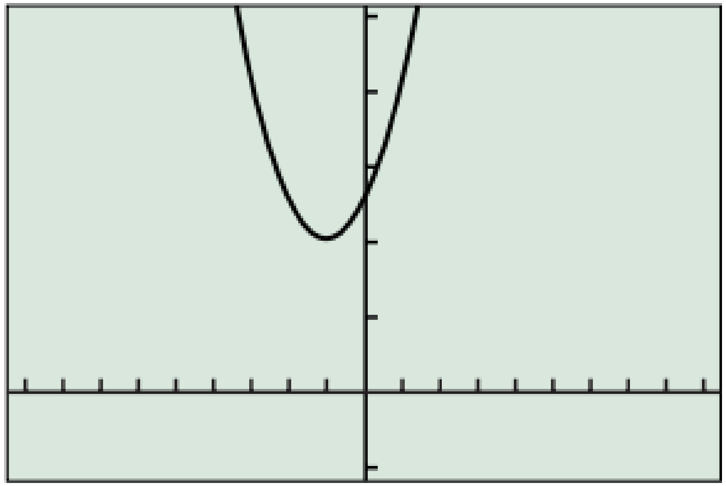
Complex Zeros and the Fundamental Theorem of Algebra



Two Major Theorems

In Section 2.3 we learned that a polynomial function of degree n has at most n real zeros. Figure 2.42 shows that the polynomial function $f(x) = x^2 + 2x + 5$ of degree 2 has no real zeros. (Why?) A little arithmetic, however, shows that the complex number $-1 + 2i$ is a zero of f:

$$\begin{aligned} f(-1 + 2i) &= (-1 + 2i)^2 + 2(-1 + 2i) + 5 \\ &= (-3 - 4i) + (-2 + 4i) + 5 \\ &= 0 + 0i \\ &= 0 \end{aligned}$$

THEOREM Fundamental Theorem of Algebra

A polynomial function of degree n has n complex zeros (real and nonreal). Some of these zeros may be repeated.

Fundamental Polynomial Connections in the Complex Case

The following statements about a polynomial function f are equivalent if k is a complex number:

1. $x = k$ is a solution (or root) of the equation $f(x) = 0$.

2. k is a zero of the function f.

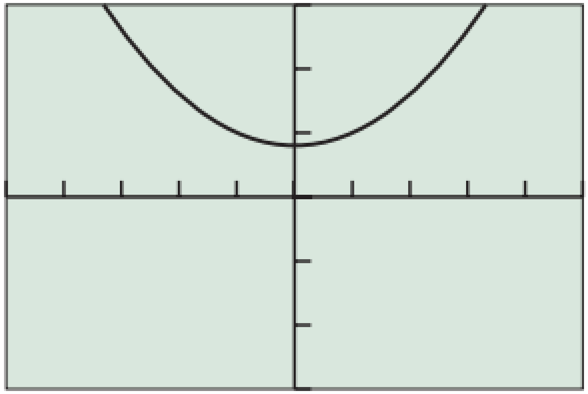
3. $x - k$ is a factor of $f(x)$.

EXAMPLE 1 Exploring Fundamental Polynomial Connections

Write the polynomial function in standard form, and identify the zeros of the function and the x-intercepts of its graph.

(a) $f(x) = (x - 2i)(x + 2i)$

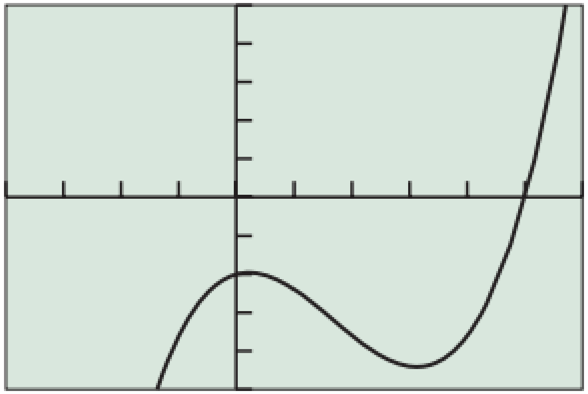
(a) The quadratic function $f(x) = (x - 2i)(x + 2i) = x^2 + 4$ has two zeros: $x = 2i$ and $x = -2i$. Because the zeros are not real, the graph of f has no x-intercepts.



(b) $f(x) = (x - 5)(x - \sqrt{2}i)(x + \sqrt{2}i)$

(b) The cubic function

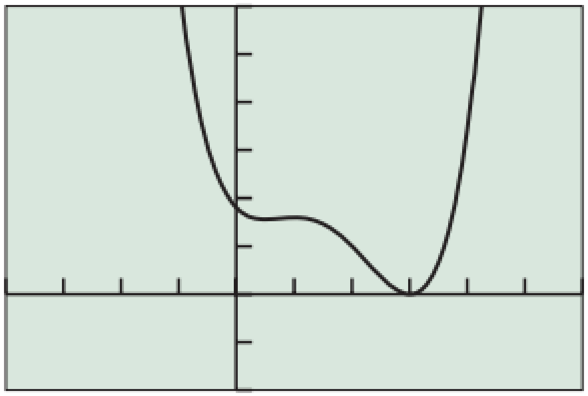
$$
\begin{aligned}
f(x) &= (x - 5)(x - \sqrt{2}i)(x + \sqrt{2}i) \\
&= (x - 5)(x^2 + 2) \\
&= x^3 - 5x^2 + 2x - 10
\end{aligned}
$$



(c) $f(x) = (x - 3)(x - 3)(x - i)(x + i)$

(c) The quartic function

$$
\begin{aligned}
f(x) &= (x - 3)(x - 3)(x - i)(x + i) \\
&= (x^2 - 6x + 9)(x^2 + 1) \\
&= x^4 - 6x^3 + 10x^2 - 6x + 9
\end{aligned}
$$



THEOREM Complex Conjugate Zeros

Suppose that $f(x)$ is a polynomial function with *real coefficients*. If a and b are real numbers with $b \neq 0$ and $a + bi$ is a zero of $f(x)$, then its complex conjugate $a - bi$ is also a zero of $f(x)$.

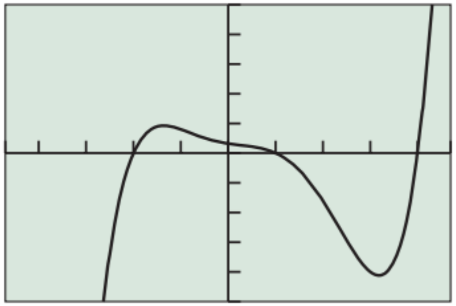
EXAMPLE 2 Finding a Polynomial from Given Zeros

Write a polynomial function of minimum degree in standard form with real coefficients whose zeros include -3, 4, and $2 - i$.

$$\begin{aligned} f(x) &= (x + 3)(x - 4)[x - (2 - i)][x - (2 + i)] \\ &= (x^2 - x - 12)(x^2 - 4x + 5) \\ &= x^4 - 5x^3 - 3x^2 + 43x - 60 \end{aligned}$$

EXAMPLE 4 Factoring a Polynomial with Complex Zeros

Find all zeros of $f(x) = x^5 - 3x^4 - 5x^3 + 5x^2 - 6x + 8$, and write $f(x)$ in its linear factorization.



EXAMPLE 5 Finding Complex Zeros

The complex number $z = 1 - 2i$ is a zero of $f(x) = 4x^4 + 17x^2 + 14x + 65$. Find the remaining zeros of $f(x)$, and write it in its linear factorization.

$$
\begin{array}{r|rrrrr}
1-2i & 4 & 0 & 17 & 14 & 65 \\
 & & 4-8i & -12-16i & -27-26i & -65 \\
\hline
 & 4 & 4-8i & 5-16i & -13-26i & 0
\end{array}
$$

$$
\begin{array}{r|rrrr}
1+2i & 4 & 4-8i & 5-16i & -13-26i \\
 & & 4+8i & 8+16i & 13+26i \\
\hline
 & 4 & 8 & 13 & 0
\end{array}
$$

Finally, we use the quadratic formula to find the two zeros of $4x^2 + 8x + 13$:

$$x = \frac{-8 \pm \sqrt{64 - 208}}{8}$$

$$= \frac{-8 \pm \sqrt{-144}}{8}$$

$$= \frac{-8 \pm 12i}{8}$$

$$= -1 \pm \frac{3}{2}i$$