Graphs of Rational Functions

Rational Functions

Rational functions are ratios (or quotients) of polynomial functions.

DEFINITION Rational Functions

Let f and g be polynomial functions with $g(x) \neq 0$. Then the function given by

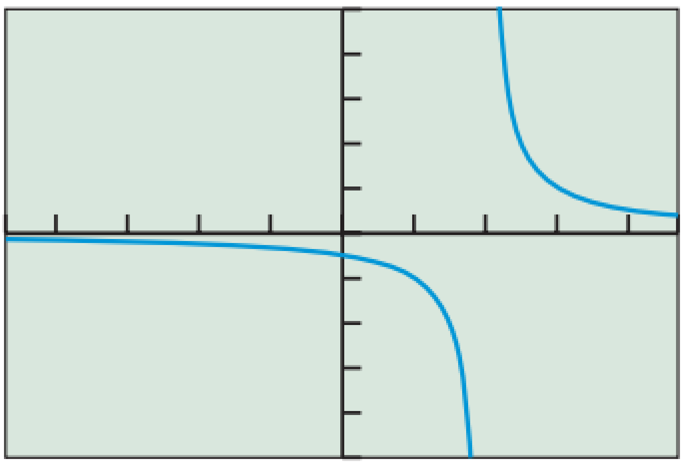
$$r(x) = \frac{f(x)}{g(x)}$$

is a **rational function**.

EXAMPLE 1 Finding the Domain of a Rational Function

Find the domain of f and use limits to describe its behavior at value(s) of x not in its domain.

$$f(x) = \frac{1}{x - 2}$$



X	Y1
2	ERROR
1.99	–100
1.98	–50
1.97	–33.33
1.96	–25
1.95	–20
1.94	–16.67

Y1 = 1/(X–2)

X	Y1
2	ERROR
2.01	100
2.02	50
2.03	33.333
2.04	25
2.05	20
2.06	16.667

Y1 = 1/(X–2)

$$\lim_{x \to 2^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \infty.$$

EXAMPLE 2 Transforming the Reciprocal Function

Describe how the graph of the given function can be obtained by transforming the graph of the reciprocal function $f(x) = 1/x$. Identify the horizontal and vertical asymptotes and use limits to describe the corresponding behavior. Sketch the graph of the function.

(a) $g(x) = \dfrac{2}{x+3}$

(b) $h(x) = \dfrac{3x-7}{x-2}$

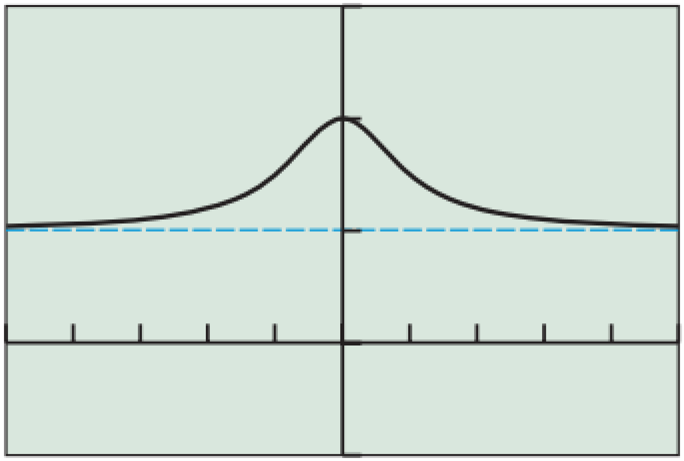
$$2\left(\frac{1}{x+3}\right) = 2f(x+3)$$

$$\begin{array}{r} 3 \\ x-2\overline{)3x-7} \\ \underline{3x-6} \\ -1 \end{array}$$

$$3 - \frac{1}{x-2} = -f(x-2) + 3.$$

EXAMPLE 3 Finding Asymptotes

Find the horizontal and vertical asymptotes of $f(x) = (x^2 + 2)/(x^2 + 1)$. Use limits to describe the corresponding behavior of f.



$$1 + \frac{1}{x^2 + 1}$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 1,$$

Graph of a Rational Function

The graph of $y = f(x)/g(x) = (a_n x^n + \cdots)/(b_m x^m + \cdots)$ has the following characteristics:

1. **End behavior asymptote**:

   If $n < m$, the end behavior asymptote is the horizontal asymptote $y = 0$.

   If $n = m$, the end behavior asymptote is the horizontal asymptote $y = a_n/b_m$.

   If $n > m$, the end behavior asymptote is the quotient polynomial function $y = q(x)$, where $f(x) = g(x)q(x) + r(x)$. There is no horizontal asymptote.

2. **Vertical asymptotes**: These occur at the zeros of the denominator, provided that the zeros are not also zeros of the numerator of equal or greater multiplicity.

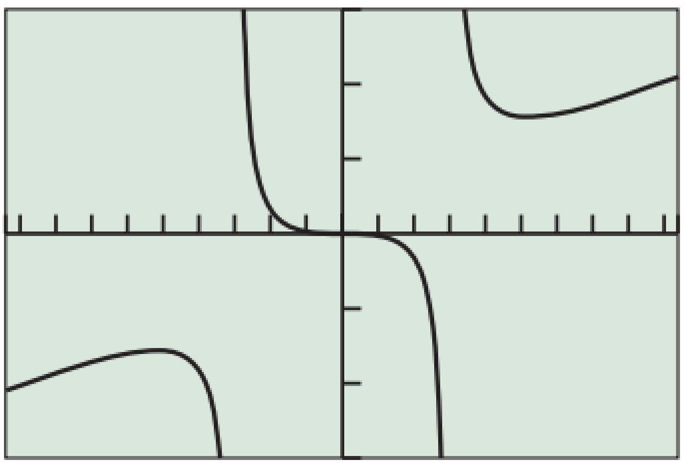
3. **x-intercepts**: These occur at the zeros of the numerator, which are not also zeros of the denominator.

4. **y-intercept**: This is the value of $f(0)$, if defined.

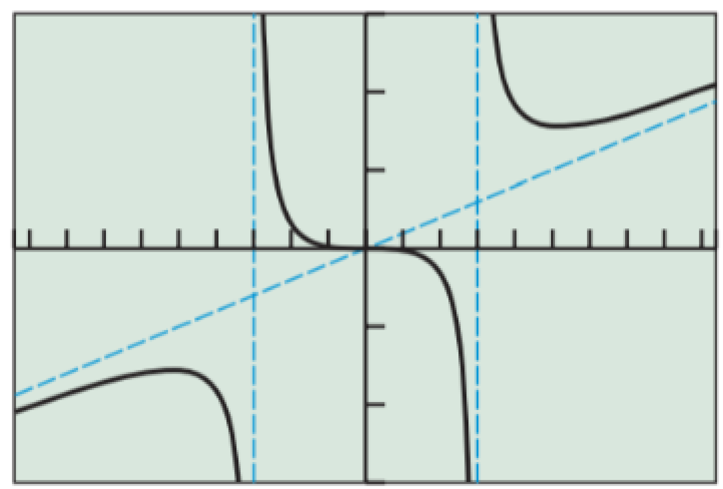
EXAMPLE 4 Graphing a Rational Function

Find the asymptotes and intercepts of the function $f(x) = x^3/(x^2 - 9)$ and graph the function.

$$f(x) = \frac{x^3}{x^2 - 9} = x + \frac{9x}{x^2 - 9}$$



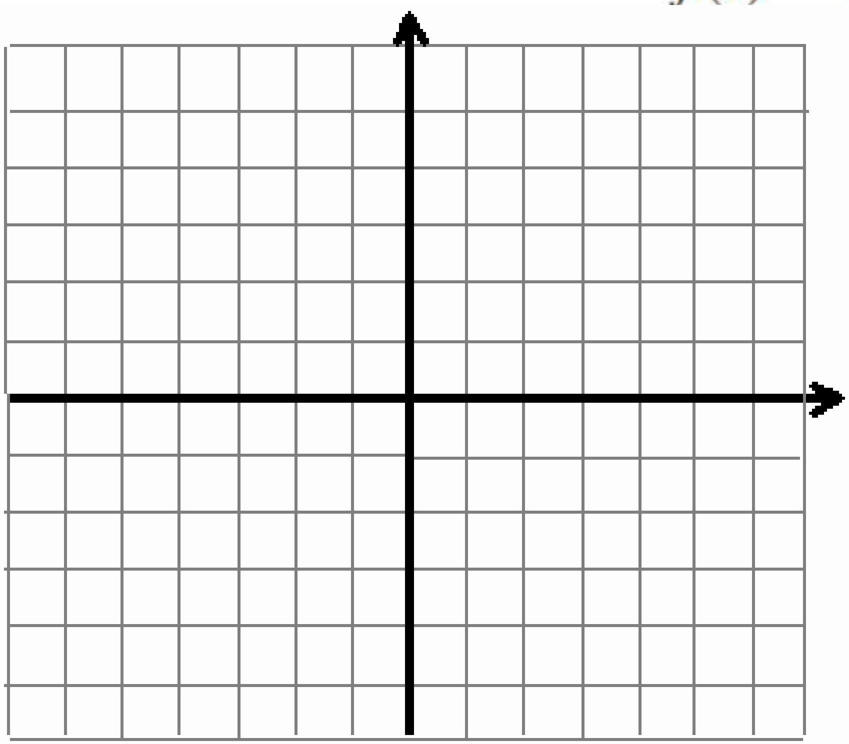
If the end behavior asymptote of a rational function is a slant line, we call it a **slant asymptote**



EXAMPLE 5 Analyzing the Graph of a Rational Function

Find the intercepts, asymptotes, use limits to describe the behavior at the vertical asymptotes, and analyze and draw the graph of the rational function

$$f(x) = \frac{x-1}{x^2 - x - 6}.$$

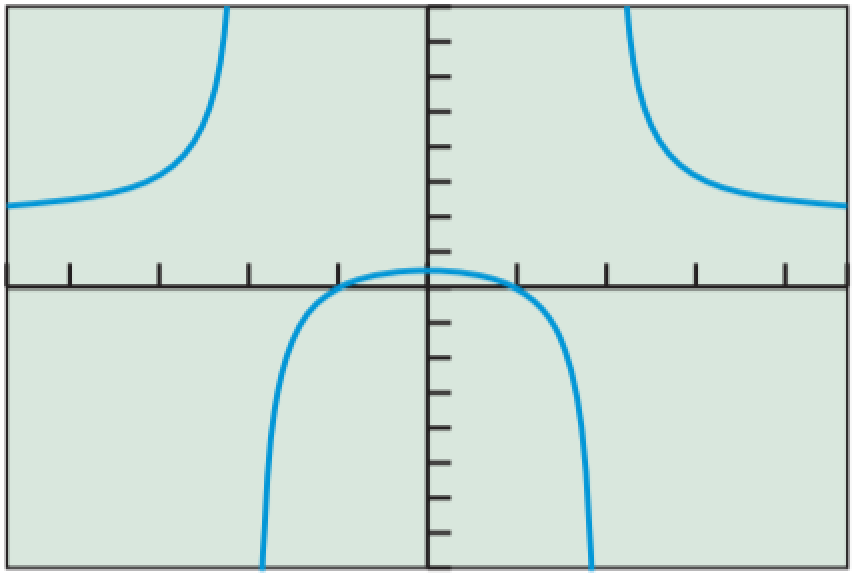


$\lim_{x \to -2^-} f(x) = -\infty$, $\lim_{x \to -2^+} f(x) = \infty$, $\lim_{x \to 3^-} f(x) = -\infty$, and $\lim_{x \to 3^+} f(x) = \infty$.

EXAMPLE 6 Analyzing the Graph of a Rational Function

Find the intercepts, analyze, and draw the graph of the rational function

$$f(x) = \frac{2x^2 - 2}{x^2 - 4}.$$



EXAMPLE 7 Finding an End-Behavior Asymptote

Find the end-behavior asymptote of

$$f(x) = \frac{x^3 - 3x^2 + 3x + 1}{x - 1}$$