Solving Inequalities in One Variable

Polynomial Inequalities

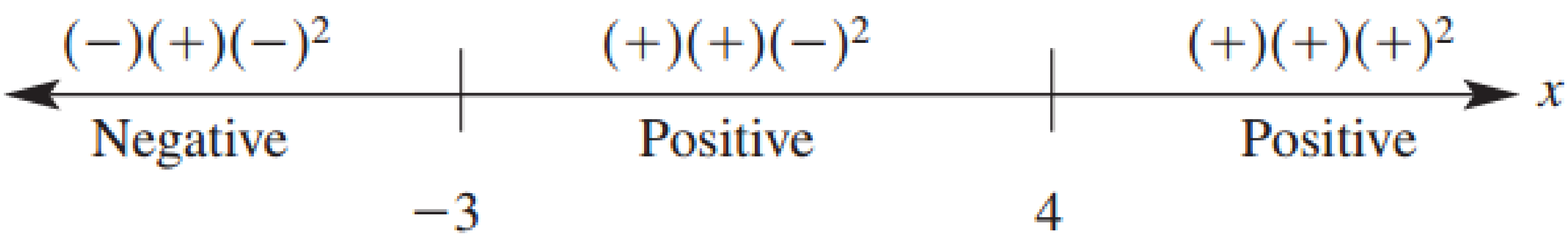
A polynomial inequality takes the form $f(x) > 0$, $f(x) \geq 0$, $f(x) < 0$, $f(x) \leq 0$ or $f(x) \neq 0$, where $f(x)$ is a polynomial. There is a fundamental connection between inequalities and the positive or negative sign of the corresponding expression $f(x)$:

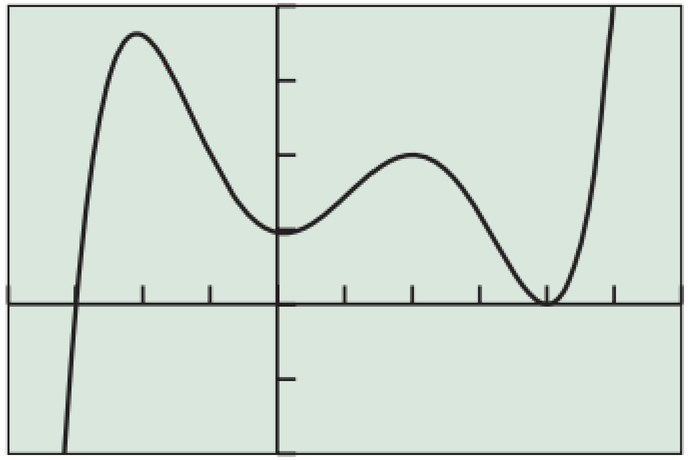
- To solve $f(x) > 0$ is to find the values of x that make $f(x)$ *positive.*
- To solve $f(x) < 0$ is to find the values of x that make $f(x)$ *negative.*

EXAMPLE 1 Finding where a Polynomial is Zero, Positive, or Negative

Let $f(x) = (x + 3)(x^2 + 1)(x - 4)^2$. Determine the real number values of x that cause $f(x)$ to be **(a)** zero, **(b)** positive, **(c)** negative.

This verbal reasoning process is aided by the following **sign chart**, which shows the x-axis as a number line with the real zeros displayed as the locations of potential sign change and the factors displayed with their sign value in the corresponding interval:





[–4, 6] by [–100, 200]

FIGURE 2.64 The graph of $f(x) = (x + 3)(x^2 + 1)(x - 4)^2$. (Example 1)

- The solution of $(x + 3)(x^2 + 1)(x - 4)^2 > 0$ is $(-3, 4) \cup (4, \infty)$.
- The solution of $(x + 3)(x^2 + 1)(x - 4)^2 \geq 0$ is $[-3, \infty)$.
- The solution of $(x + 3)(x^2 + 1)(x - 4)^2 < 0$ is $(-\infty, -3)$.
- The solution of $(x + 3)(x^2 + 1)(x - 4)^2 \leq 0$ is $(-\infty, -3] \cup \{4\}$.

EXAMPLE 2 Solving a Polynomial Inequality Analytically

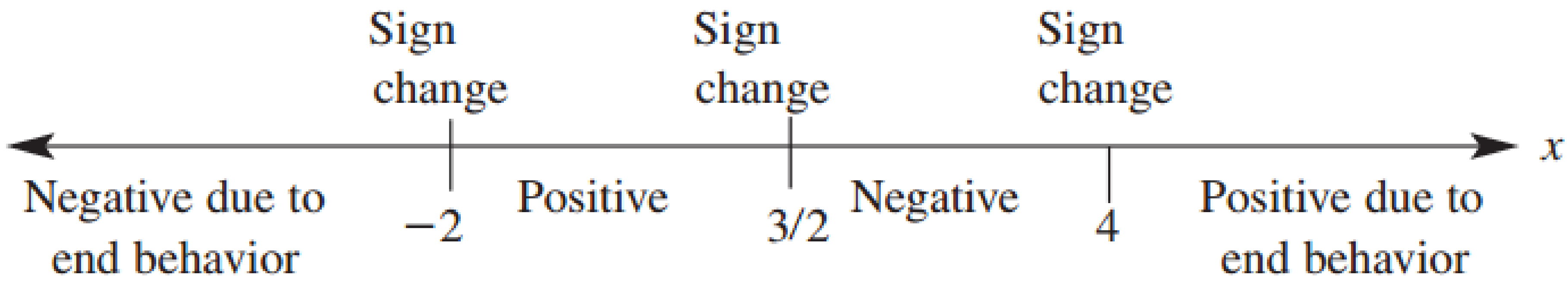
Solve $2x^3 - 7x^2 - 10x + 24 > 0$ analytically.

The Rational Zeros Theorem yields $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24, \pm \frac{1}{2}, \pm \frac{3}{2}$.

$$
\begin{array}{r|rrrr}
4 & 2 & -7 & -10 & 24 \\
  &   & 8 & 4 & -24 \\
\hline
  & 2 & 1 & -6 & 0
\end{array}
$$

$$
\begin{aligned}
f(x) &= 2x^3 - 7x^2 - 10x + 24 \\
&= (x - 4)(2x^2 + x - 6) \\
&= (x - 4)(2x - 3)(x + 2)
\end{aligned}
$$

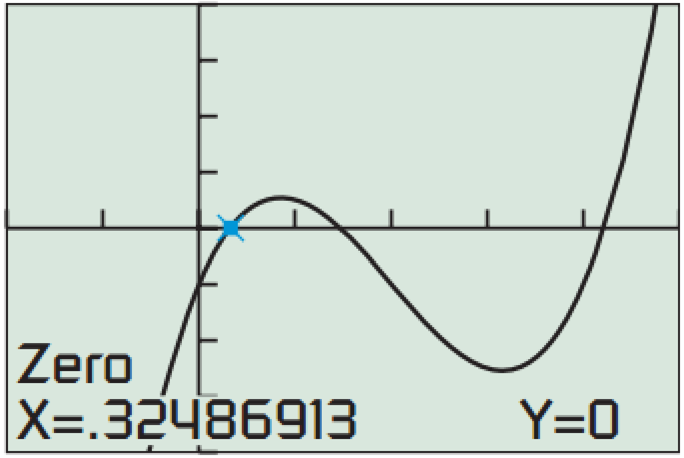
Our analysis yields the following sign chart:



EXAMPLE 3 Solving a Polynomial Inequality Graphically

Solve $x^3 - 6x^2 \leq 2 - 8x$ graphically.

First we rewrite the inequality as $x^3 - 6x^2 + 8x - 2 \leq 0$.

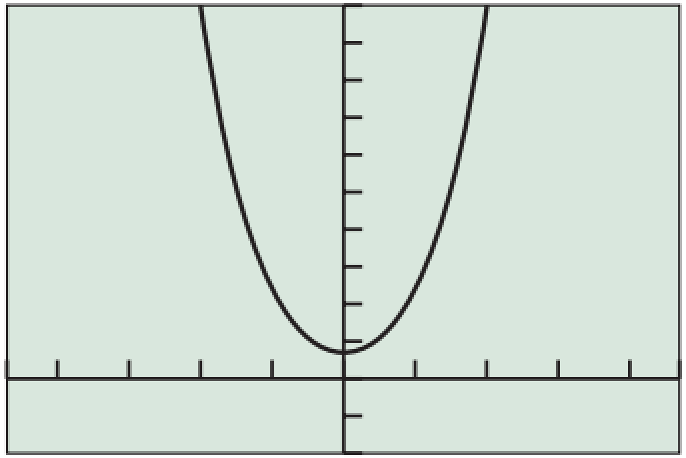


[–2, 5] by [–8, 8]

EXAMPLE 4 Solving a Polynomial Inequality with Unusual Answers

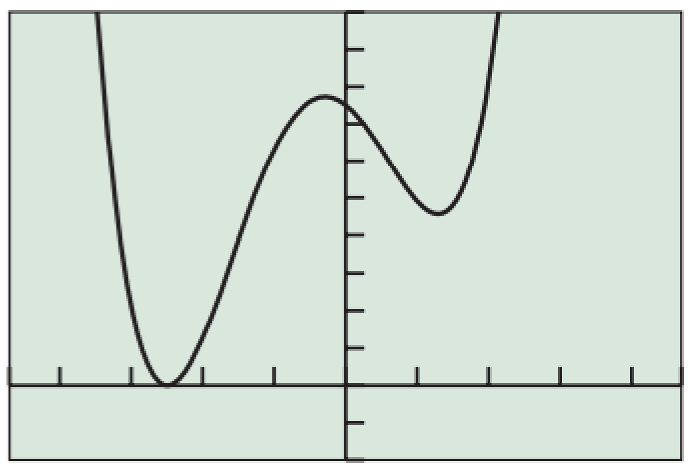
(a) The inequalities associated with the strictly positive polynomial function $f(x) = (x^2 + 7)(2x^2 + 1)$ have unusual solution sets. We use Figure 2.67a as a guide to solving the inequalities:

- The solution of $(x^2 + 7)(2x^2 + 1) > 0$ is $(-\infty, \infty)$, all real numbers.
- The solution of $(x^2 + 7)(2x^2 + 1) \geq 0$ is also $(-\infty, \infty)$.
- The solution set of $(x^2 + 7)(2x^2 + 1) < 0$ is empty. We say an inequality of this sort has no solution.
- The solution set of $(x^2 + 7)(2x^2 + 1) \leq 0$ is also empty.



(b) The inequalities associated with the nonnegative polynomial function $g(x) = (x^2 - 3x + 3)(2x + 5)^2$ also have unusual solution sets. We use Figure 2.67b as a guide to solving the inequalities:

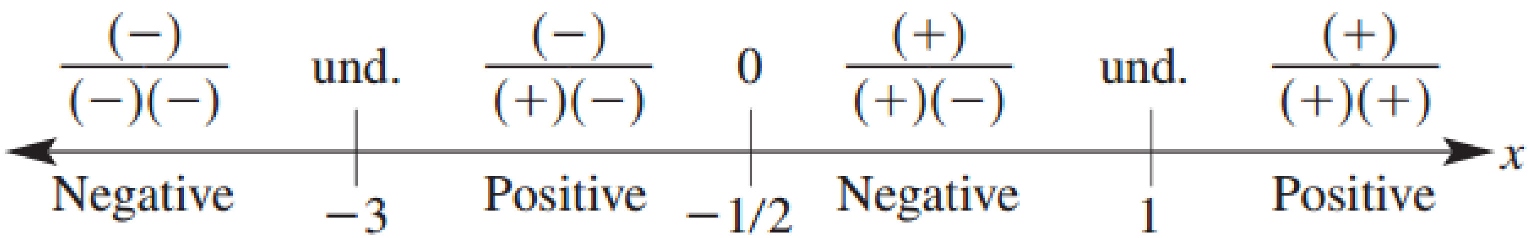
- The solution of $(x^2 - 3x + 3)(2x + 5)^2 > 0$ is $(-\infty, -5/2) \cup (-5/2, \infty)$, all real numbers except $x = -5/2$, the lone real zero of g.
- The solution of $(x^2 - 3x + 3)(2x + 5)^2 \geq 0$ is $(-\infty, \infty)$, all real numbers.
- The solution set of $(x^2 - 3x + 3)(2x + 5)^2 < 0$ is empty.
- The solution of $(x^2 - 3x + 3)(2x + 5)^2 \leq 0$ is the single number $x = -5/2$.

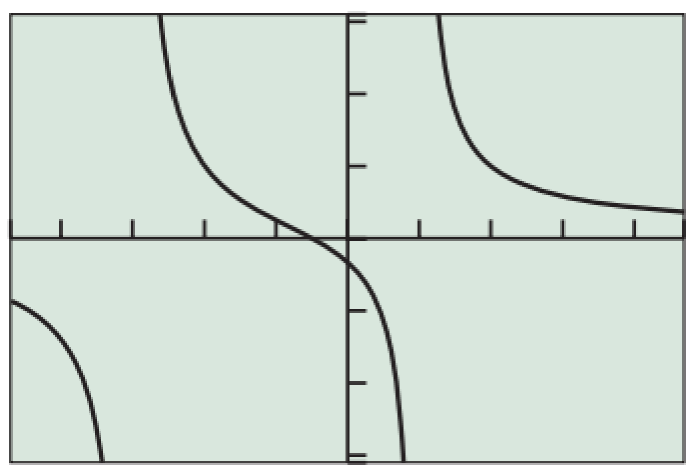


EXAMPLE 5 Creating a Sign Chart for a Rational Function

Let $r(x) = (2x + 1)/((x + 3)(x - 1))$. Determine the values of x that cause $r(x)$ to be **(a)** zero, **(b)** undefined. Then make a sign chart to determine the values of x that cause $r(x)$ to be **(c)** positive, **(d)** negative.

Analyzing the factors of the numerator and denominator yields:





(a) The real zeros of $r(x)$ are the real zeros of the numerator $2x + 1$. So $r(x)$ is zero if $x = -1/2$.

(b) $r(x)$ is undefined when the denominator $(x + 3)(x - 1) = 0$. So $r(x)$ is undefined if $x = -3$ or $x = 1$.

(c) So $r(x)$ is positive if $-3 < x < -1/2$ or $x > 1$, and the solution of $(2x + 1)/((x + 3)(x - 1)) > 0$ is $(-3, -1/2) \cup (1, \infty)$.

(d) Similarly, $r(x)$ is negative if $x < -3$ or $-1/2 < x < 1$, and the solution of $(2x + 1)/((x + 3)(x - 1)) < 0$ is $(-\infty, -3) \cup (-1/2, 1)$.