Logistic Functions and Their Graphs

Exponential growth is *unrestricted*. An exponential growth function increases at an ever increasing rate and is not bounded above. In many growth situations, however, there is a limit to the possible growth. A plant can only grow so tall. The number of goldfish in an aquarium is limited by the size of the aquarium. In such situations the growth often begins in an exponential manner, but the growth eventually slows and the graph levels out. The associated growth function is bounded both below and above by horizontal asymptotes.

DEFINITION Logistic Growth Functions

Let a, b, c, and k be positive constants, with $b < 1$. A **logistic growth function** in x is a function that can be written in the form

$$f(x) = \frac{c}{1 + a \cdot b^x} \text{ or } f(x) = \frac{c}{1 + a \cdot e^{-kx}}$$

where the constant c is the **limit to growth**.

By setting $a = c = k = 1$, we obtain the **logistic function**

$$f(x) = \frac{1}{1 + e^{-x}}.$$

BASIC FUNCTION The Logistic Function

$$f(x) = \frac{1}{1 + e^{-x}}$$

Domain: All reals
Range: (0, 1)
Continuous
Increasing for all x
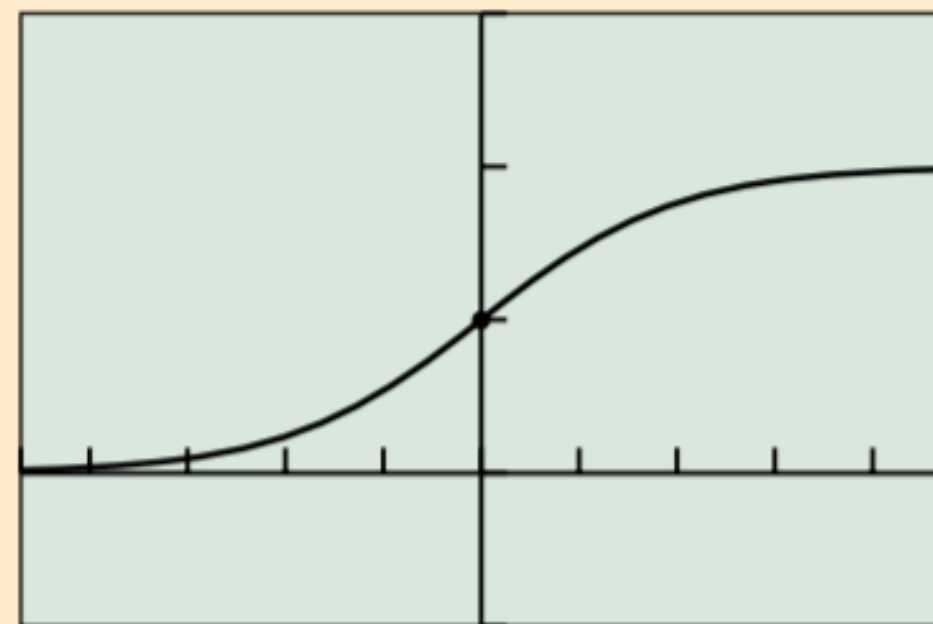Symmetric about (0, 1/2), but neither even nor odd
Bounded below and above
No local extrema
Horizontal asymptotes: $y = 0$ and $y = 1$
No vertical asymptotes
End behavior: $\lim\limits_{x \to -\infty} f(x) = 0$ and $\lim\limits_{x \to \infty} f(x) = 1$

[–4.7, 4.7] by [–0.5, 1.5]

EXAMPLE 6   Graphing Logistic Growth Functions

Graph the function. Find the y-intercept and the horizontal asymptotes.

(a) $f(x) = \dfrac{8}{1 + 3 \cdot 0.7^x}$

(b) $g(x) = \dfrac{20}{1 + 2e^{-3x}}$

EXAMPLE 7 Modeling San Jose's Population

Using the data in Table 3.5 and assuming the growth is exponential, when will the population of San Jose surpass 1 million persons?

Table 3.5 The Population of San Jose, California

Year	Population
1990	782,248
2000	895,193

EXAMPLE 8 Modeling Dallas's Population

Based on recent census data, a logistic model for the population of Dallas, t years after 1900, is as follows:

$$P(t) = \frac{1{,}301{,}642}{1 + 21.602e^{-0.05054t}}$$

According to this model, when was the population 1 million?