Proving Trigonometric Identities

A Proof Strategy

We now arrive at the best opportunity in the precalculus curriculum for you to try your hand at constructing analytic proofs: trigonometric identities. Some are easy and some can be quite challenging, but in every case the *identity itself* frames your work with a beginning and ending. The proof consists of filling in what lies between.

EXAMPLE 1 Proving an Algebraic Identity

Prove the algebraic identity $\dfrac{x^2-1}{x-1} - \dfrac{x^2-1}{x+1} = 2.$

$$
\begin{aligned}
\frac{x^2-1}{x-1} - \frac{x^2-1}{x+1} &= \frac{(x+1)(x-1)}{x-1} - \frac{(x+1)(x-1)}{x+1} && \text{Factoring difference of squares} \\
&= (x+1)\left(\frac{x-1}{x-1}\right) - (x-1)\left(\frac{x+1}{x+1}\right) && \text{Algebraic manipulation} \\
&= (x+1)(1) - (x-1)(1) && \text{Reducing fractions} \\
&= x + 1 - x + 1 && \text{Algebraic manipulation} \\
&= 2
\end{aligned}
$$

General Strategies I

1. The proof begins with the expression on one side of the identity.
2. The proof ends with the expression on the other side.
3. The proof in between consists of showing a sequence of expressions, each one easily seen to be equivalent to its preceding expression.

EXAMPLE 2 Proving an Identity

Prove the identity: $\tan x + \cot x = \sec x \csc x$.

General Strategies II

1. Begin with the more complicated expression and work toward the less complicated expression.
2. If no other move suggests itself, convert the entire expression to one involving sines and cosines.
3. Combine fractions by combining them over a common denominator.

EXAMPLE 3 Identifying and Proving an Identity

Match the function

$$f(x) = \frac{1}{\sec x - 1} + \frac{1}{\sec x + 1}$$

with one of the following. Then confirm the match with a proof.

(i) $2 \cot x \csc x$ **(ii)** $\frac{1}{\sec x}$

EXAMPLE 4 Setting up a Difference of Squares

Prove the identity: $\cos t/(1 - \sin t) = (1 + \sin t)/\cos t$.

General Strategies III

1. Use the algebraic identity $(a + b)(a - b) = a^2 - b^2$ to set up applications of the Pythagorean identities.
2. Always be mindful of the "target" expression, and favor manipulations that bring you closer to your goal.

EXAMPLE 5 Working from Both Sides

Prove the identity: $\cot^2 u/(1 + \csc u) = (\cot u)(\sec u - \tan u)$.