Sum and Difference Identities

Cosine of a Difference

There is a powerful instinct in all of us to believe that all functions obey the following law of additivity:

$$f(u+v) = f(u) + f(v).$$

In fact, very few do. If there were a Hall of Fame for algebraic blunders, the following would probably be the first two inductees:

$$(u+v)^2 = u^2 + v^2$$

$$\sqrt{u+v} = \sqrt{u} + \sqrt{v}$$

We could also show easily that

$$\cos(u - v) \neq \cos(u) - \cos(v) \quad \text{and} \quad \sin(u - v) \neq \sin(u) - \sin(v).$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v.$$

EXAMPLE 1 Using the Cosine-of-a-Difference Identity

Find the exact value of cos 15° without using a calculator.

$$\cos 15^\circ = \cos(45^\circ - 30^\circ)$$

Cosine of a Sum

Now that we have the formula for the cosine of a difference, we can get the formula for the cosine of a sum almost for free by using the odd-even identities.

$$\cos(u + v) = \cos(u - (-v))$$

Cosine of a Sum or Difference

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

(Note the sign switch in either case.)

EXAMPLE 2 Confirming Cofunction Identities

Prove the identities **(a)** $\cos((\pi/2) - x) = \sin x$ and **(b)** $\sin((\pi/2) - x) = \cos x$.

(a) $$\cos\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{\pi}{2}\right)\sin x$$ Cosine sum identity

$$= 0 \cdot \cos x + 1 \cdot \sin x$$

$$= \sin x$$

(b) $$\sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) = \cos(0 + x) = \cos x$$

Sine of a Difference or Sum

We can use the cofunction identities in Example 2 to get the formula for the sine of a sum from the formula for the cosine of a difference.

$$\sin(u+v) = \cos\left(\frac{\pi}{2} - (u+v)\right) \qquad \text{Cofunction identity}$$

Then we can use the odd-even identities to get the formula for the sine of a difference from the formula for the sine of a sum.

$$\sin(u - v) = \sin(u + (-v))$$ A little algebra

Sine of a Sum or Difference

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

(Note that the sign does *not* switch in either case.)

EXAMPLE 3 Using the Sum/Difference Formulas

Write each of the following expressions as the sine or cosine of an angle.

(a) $\sin 22° \cos 13° + \cos 22° \sin 13°$

(b) $\cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4}$

(c) $\sin x \sin 2x - \cos x \cos 2x$

SOLUTION The key in each case is recognizing which formula applies. (Indeed, the real purpose of such exercises is to help you remember the formulas.)

If one of the angles in a sum or difference is a quadrantal angle (that is, a multiple of $90°$ or of $\pi/2$ radians), then the sum-difference identities yield single-termed expressions. Since the effect is to reduce the complexity, the resulting identity is called a **reduction formula**.

EXAMPLE 4 Proving Reduction Formulas

Prove the reduction formulas:

(a) $\sin(x + \pi) = -\sin x$

(b) $\cos\left(x + \frac{3\pi}{2}\right) = \sin x$

Tangent of a Difference or Sum

We can derive a formula for tan $(u \pm v)$ directly from the corresponding formulas for sine and cosine, as follows:

$$\tan(u \pm v) = \frac{\sin(u \pm v)}{\cos(u \pm v)} = \frac{\sin u \cos v \pm \cos u \sin v}{\cos u \cos v \mp \sin u \sin v}.$$

There is also a formula for tan $(u \pm v)$ that is written entirely in terms of tangent functions:

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$