Multiple-Angle Identities

Double-Angle Identities

Double-Angle Identities

$$\sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \begin{cases} \cos^2 u - \sin^2 u \\ 2\cos^2 u - 1 \\ 1 - 2\sin^2 u \end{cases}$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

EXAMPLE 1 Proving a Double-Angle Identity

Prove the identity: $\sin 2u = 2 \sin u \cos u$.

$$\sin 2u = \sin (u + u)$$

$$= \sin u \cos u + \cos u \sin u \qquad \text{Sine of a sum } (v = u)$$

$$= 2 \sin u \cos u$$

EXAMPLE 2 **Proving an Identity**

Prove the identity: $\cos^4\theta - \sin^4\theta = \cos 2\theta$.

$$(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$$

$$1 \cdot (\cos^2\theta - \sin^2\theta)$$ Pythagorean identity

$$\cos 2\theta$$ Double-angle identity

Power-Reducing Identities

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

EXAMPLE 3 Reducing a Power of 4

Rewrite $\cos^4 x$ in terms of trigonometric functions with no power greater than 1.

$$(\cos^2 x)^2$$

$$\left(\frac{1+\cos 2x}{2}\right)^2 \qquad \text{Power-reducing identity}$$

$$\left(\frac{1+2\cos 2x+\cos^2 2x}{4}\right)$$

$$\frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{4}\left(\frac{1+\cos 4x}{2}\right) \qquad \text{Power-reducing identity}$$

$$\frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{8}+\frac{1}{8}\cos 4x$$

$$\frac{1}{8}(3+4\cos 2x+\cos 4x)$$

Half-Angle Identities

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \begin{cases} \pm\sqrt{\dfrac{1-\cos u}{1+\cos u}} \\ \dfrac{1-\cos u}{\sin u} \\ \dfrac{\sin u}{1+\cos u} \end{cases}$$

EXAMPLE 4 **Using a Double-Angle Identity**

Solve algebraically in the interval $[0, 2\pi)$: $\sin 2x = \cos x$.

$$2 \sin x \cos x = \cos x$$

$$2 \sin x \cos x - \cos x = 0$$

$$\cos x\,(2 \sin x - 1) = 0$$

$$\cos x = 0 \quad \text{or} \quad 2 \sin x - 1 = 0$$

$$\cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2}$$

$$\frac{\pi}{6}, \quad \frac{\pi}{2}, \quad \frac{5\pi}{6}, \quad \frac{3\pi}{2}$$

EXAMPLE 5 Using Half-Angle Identities

Solve $\sin^2 x = 2\sin^2 (x/2)$.

$$\sin^2 x = 2\left(\frac{1 - \cos x}{2}\right)$$ Half-angle identity

$$1 - \cos^2 x = 1 - \cos x$$ Convert to all cosines.

$$\cos x - \cos^2 x = 0$$

$$\cos x\,(1 - \cos x) = 0$$

$$\cos x = 0 \quad \text{or} \quad \cos x = 1$$

$$\cos x = 0 \quad \text{or} \quad \cos x = 1$$

$$x = \frac{\pi}{2} \quad \text{or} \quad \frac{3\pi}{2} \quad \text{or} \quad 0$$

The rest of the solutions are obtained by periodicity:

$$x = 2n\pi, \quad x = \frac{\pi}{2} + 2n\pi, \quad x = \frac{3\pi}{2} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$