Infinite Sequences

One of the most natural ways to study patterns in mathematics is to look at an ordered progression of numbers, called a **sequence**. Here are some examples of sequences:

1. 5, 10, 15, 20, 25
2. $2, 4, 8, 16, 32, \ldots, 2^k, \ldots$
3. $\left\{\frac{1}{k}: k = 1, 2, 3, \ldots\right\}$
4. $\{a_1, a_2, a_3, \ldots, a_k, \ldots\}$, which is sometimes abbreviated $\{a_k\}$

The first of these is a **finite sequence**, while the other three are **infinite sequences**. Notice that in (2) and (3) we were able to define a rule that gives the kth number in the sequence (called the ***k*th term**) as a function of k. In (4) we do not have a rule, but notice how we can use subscript notation (a_k) to identify the kth term of a "general" infinite sequence. In this sense, an infinite sequence can be thought of as a *function* that assigns a unique number (a_k) to each natural number k.

EXAMPLE 1 Defining a Sequence Explicitly

Find the first 6 terms and the 100th term of the sequence $\{a_k\}$ in which $a_k = k^2 - 1$.

EXAMPLE 2 Defining a Sequence Recursively

Find the first 6 terms and the 100th term for the sequence defined recursively by the conditions:

$$b_1 = 3$$

$$b_n = b_{n-1} + 2 \text{ for all } n > 1.$$

Limits of Infinite Sequences

Just as we were concerned with the end behavior of functions, we will also be concerned with the end behavior of sequences.

DEFINITION Limit of a Sequence

Let $\{a_n\}$ be a sequence of real numbers, and consider $\lim_{n\to\infty} a_n$. If the limit is a finite number L, the sequence **converges** and L is the **limit of the sequence**. If the limit is infinite or nonexistent, the sequence **diverges**.

EXAMPLE 3 Finding Limits of Sequences

Determine whether the sequence converges or diverges. If it converges, give the limit.

(a) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

(b) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$

(c) $2, 4, 6, 8, 10, \ldots$

(d) $-1, 1, -1, 1, \ldots, (-1)^n, \ldots$

EXAMPLE 4 Finding Limits of Sequences

Determine whether the sequence converges or diverges. If it converges, give the limit.

(a) $\left\{\dfrac{3n}{n+1}\right\}$ (b) $\left\{\dfrac{5n^2}{n^3+1}\right\}$ (c) $\left\{\dfrac{n^3+2}{n^2+n}\right\}$

Arithmetic and Geometric Sequences

There are all kinds of rules by which we can construct sequences, but two particular types of sequences dominate in mathematical applications: those in which pairs of successive terms all have a common *difference* (**arithmetic** sequences), and those in which pairs of successive terms all have a common quotient, or *ratio* (**geometric** sequences). We will take a closer look at these in this section.

DEFINITION **Arithmetic Sequence**

A sequence $\{a_n\}$ is an **arithmetic sequence** if it can be written in the form

$$\{a, a + d, a + 2d, \ldots, a + (n - 1)d, \ldots\} \text{ for some constant } d.$$

The number d is called the **common difference**.

Each term in an arithmetic sequence can be obtained recursively from its preceding term by adding d:

$$a_n = a_{n-1} + d \text{ (for all } n \geq 2).$$

EXAMPLE 5 Defining Arithmetic Sequences

For each of the following arithmetic sequences, find **(a)** the common difference, **(b)** the tenth term, **(c)** a recursive rule for the nth term, and **(d)** an explicit rule for the nth term.

(1) $-6, -2, 2, 6, 10, \ldots$

(2) $\ln 3, \ln 6, \ln 12, \ln 24, \ldots$

DEFINITION **Geometric Sequence**

A sequence $\{a_n\}$ is a **geometric sequence** if it can be written in the form

$$\{a, a \cdot r, a \cdot r^2, \ldots, a \cdot r^{n-1}, \ldots\} \text{ for some nonzero constant } r.$$

The number r is called the **common ratio**.

Each term in a geometric sequence can be obtained recursively from its preceding term by multiplying by r:

$$a_n = a_{n-1} \cdot r \text{ (for all } n \geq 2).$$

EXAMPLE 6 Defining Geometric Sequences

For each of the following geometric sequences, find **(a)** the common ratio, **(b)** the tenth term, **(c)** a recursive rule for the nth term, and **(d)** an explicit rule for the nth term.

(1) $3, 6, 12, 24, 48, \ldots$

(2) $10^{-3}, 10^{-1}, 10^{1}, 10^{3}, 10^{5}, \ldots$