Infinite Sequences

One of the most natural ways to study patterns in mathematics is to look at an ordered progression of numbers, called a **sequence**. Here are some examples of sequences:

1. 5, 10, 15, 20, 25 $5k$
2. $2, 4, 8, 16, 32, \ldots, 2^k, \ldots$
3. $\left\{\frac{1}{k}: k = 1, 2, 3, \ldots\right\}$
4. $\{a_1, a_2, a_3, \ldots, a_k, \ldots\}$, which is sometimes abbreviated $\{a_k\}$

The first of these is a **finite sequence**, while the other three are **infinite sequences**. Notice that in (2) and (3) we were able to define a rule that gives the kth number in the sequence (called the ***k*th term**) as a function of k. In (4) we do not have a rule, but notice how we can use subscript notation (a_k) to identify the kth term of a "general" infinite sequence. In this sense, an infinite sequence can be thought of as a *function* that assigns a unique number (a_k) to each natural number k.

EXAMPLE 1 Defining a Sequence Explicitly

Find the first 6 terms and the 100th term of the sequence $\{a_k\}$ in which $a_k = k^2 - 1$.

$a_1 = 1^2 - 1 = 0$

$a_2 = 2^2 - 1 = 3$

$a_3 = 3^2 - 1 = 8$

$a_4 = 4^2 - 1 = 15$

$a_5 = 5^2 - 1 = 24$

$a_6 = 6^2 - 1 = 35$

0, 3, 8, 15, 24, 35

$a_{100} = 100^2 - 1 = 9999$

EXAMPLE 2 **Defining a Sequence Recursively**

Find the first 6 terms and the 100th term for the sequence defined recursively by the conditions:

starting

using previous

$$b_1 = 3$$

$$b_n = b_{n-1} + 2 \text{ for all } n > 1.$$

$3, 5, 7, 9, 11, 13 \ldots$

$b_2 = b_{2-1} + 2 = b_1 + 2 = 3 + 2 = 5$

$b_3 = b_{3-1} + 2 = b_2 + 2 = 5 + 2 = 7$

$b_4 = \quad = b_3 + 2 = 7 + 2 = 9$

$b_5 = \quad = b_4 + 2 = 9 + 2 = 11$

$b_6 = \quad = b_5 + 2 = 11 + 2 = 13$

$a_{100} = 3 + 99(2)$

$a_{100} = 3 + 198 = 201$

EXAMPLE 3 Finding Limits of Sequences

Determine whether the sequence converges or diverges. If it converges, give the limit.

(a) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ $\lim_{n\to\infty} \frac{1}{n} = \boxed{0}$, converges

(b) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$ $\frac{n+1}{n}$ $\lim_{n\to\infty} \frac{n+1}{n} = 1 + \cancel{\frac{1}{n}}^{0} = \boxed{1}$ converges

(c) $2, 4, 6, 8, 10, \ldots$ $2n$ $\lim_{n\to\infty} 2n$ DNE, ∞ Diverges

(d) $-1, 1, -1, 1, \ldots, (-1)^n, \ldots$ $\lim_{n\to\infty} (-1)^n =$ DNE, Diverges

Oscillating

EXAMPLE 4 **Finding Limits of Sequences**

Determine whether the sequence converges or diverges. If it converges, give the limit.

(a) $\left\{\dfrac{3n}{n+1}\right\}$ (b) $\left\{\dfrac{5n^2}{n^3+1}\right\}$ (c) $\left\{\dfrac{n^3+2}{n^2+n}\right\}$

Converges to 3

Converges to zero

Diverges DNE

Arithmetic and Geometric Sequences

There are all kinds of rules by which we can construct sequences, but two particular types of sequences dominate in mathematical applications: those in which pairs of successive terms all have a common *difference* (**arithmetic** sequences), and those in which pairs of successive terms all have a common quotient, or *ratio* (**geometric** sequences). We will take a closer look at these in this section.

5, 10, 15, 20, 25

$d = 5$

2, 4, 8, 16, 32 .....

$r = 2$

DEFINITION **Arithmetic Sequence**

A sequence $\{a_n\}$ is an **arithmetic sequence** if it can be written in the form

$$\{a, a + d, a + 2d, \ldots, a + (n - 1)d, \ldots\} \text{ for some constant } d.$$

The number d is called the **common difference**.

$a_n = a_1 + (n-1)d$

Each term in an arithmetic sequence can be obtained recursively from its preceding term by adding d:

$$a_n = a_{n-1} + d \text{ (for all } n \geq 2).$$

previous

EXAMPLE 5 Defining Arithmetic Sequences

For each of the following arithmetic sequences, find **(a)** the common difference, **(b)** the tenth term, **(c)** a recursive rule for the *n*th term, and **(d)** an explicit rule for the *n*th term.

(1) $-6, -2, 2, 6, 10, \ldots$

(2) $\ln 3, \ln 6, \ln 12, \ln 24, \ldots$

a.) 4

b.) $a_{10} = -6 + (10-1)4 = -6 + 36 = 30$

c.) $a_n = a_{n-1} + 4$

d.) $a_n = -6 + (n-1)4$

$a_n = -6 + 4n - 4 = \boxed{4n - 10}$

2 $d = \ln 6 - \ln 3 = \ln \frac{6}{3}$

a.) $d = \ln 2$

b.) $a_{10} = \ln 3 + (10-1)\ln 2$

$= \ln 3 + 9\ln 2 = \ln 3 + \ln 2^9$

$a_{10} = \ln(3 \cdot 2^9) = \ln 1536$

c.) $a_n = a_{n-1} + \ln 2$

d.) $a_n = \ln 3 + (n-1)\ln 2$

$= \ln 3 + \ln 2^{n-1}$

$a_n = \ln(3 \cdot 2^{n-1})$

DEFINITION **Geometric Sequence**

A sequence $\{a_n\}$ is a **geometric sequence** if it can be written in the form

$$\{a, a \cdot r, a \cdot r^2, \ldots, a \cdot r^{n-1}, \ldots\} \text{ for some nonzero constant } r.$$

The number r is called the **common ratio**.

$a_n = a_1 \cdot r^{n-1}$

Each term in a geometric sequence can be obtained recursively from its preceding term by multiplying by r:

$$a_n = a_{n-1} \cdot r \text{ (for all } n \geq 2).$$

EXAMPLE 6 Defining Geometric Sequences

For each of the following geometric sequences, find **(a)** the common ratio, **(b)** the tenth term, **(c)** a recursive rule for the *n*th term, and **(d)** an explicit rule for the *n*th term.

(1) 3, 6, 12, 24, 48, . . .

a.) $r = 2$

b.) $a_{10} = 3 \cdot (2)^{10-1} = 3 \cdot 2^9 = 1536$

c.) $a_n = a_{n-1}(2)$

d.) $a_n = 3(2)^{n-1}$

(2) $10^{-3}, 10^{-1}, 10^{1}, 10^{3}, 10^{5}, \ldots$

a.) $r = 10^2$

b.) $a_{10} = 10^{-3}(10^2)^{10-1} = 10^{-3} \cdot (10^2)^9$

$= 10^{-3} \cdot 10^{18}$

$a_{10} = 10^{15}$

c.) $a_n = a_{n-1}(10^2)$

d.) $a_n = 10^{-3}(10^2)^{n-1}$

$a_n = 10^{-3}(10^{2n-2}) = 10^{2n-5}$