DEFINITION Infinite Series

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots.$$

What makes series so interesting is that sometimes (as in Example 2) the sequence of **partial sums**, all of which are true sums, approaches a finite limit S:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) = S.$$

In this case we say that the series **converges** to S, and it makes sense to define S as the **sum of the infinite series**. In sigma notation,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = S.$$

If the limit of partial sums does not exist, then the series **diverges** and has no sum.

EXAMPLE 3 Looking at Limits of Partial Sums

For each of the following series, find the first five terms in the sequence of partial sums. Which of the series appear to converge?

Converging

$\frac{1}{10}$

(a) $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$ $S_1 = 0.1$, $S_2 = 0.11$, $S_3 = 0.111$ $\longrightarrow .\overline{1}$

Diverging

(b) $10 + 20 + 30 + 40 + \cdots$ $S_1 = 10$, $S_2 = 30$, $S_3 = 60$ . . . . . .

(c) $1 - 1 + 1 - 1 + \cdots$

$S_1 = 1$, $S_2 = 0$, $S_3 = 1$, $S_4 = 0$ oscillating $\rightarrow$ Diverge

THEOREM **Sum of an Infinite Geometric Series**

The geometric series $\sum_{k=1}^{\infty} a \cdot r^{k-1}$ converges if and only if $|r| < 1$. If it does converge, the sum is $a/(1 - r)$.

$$\frac{a_1}{1-r} = \frac{0.1}{1-0.1} = \frac{0.1}{0.9} = \boxed{\frac{1}{9}}$$

EXAMPLE 4 Summing Infinite Geometric Series

Determine whether the series converges. If it converges, give the sum.

(a) $\sum_{k=1}^{\infty} 3(0.75)^{k-1}$ $r = .75$

(b) $\sum_{n=0}^{\infty} \left(-\frac{4}{5}\right)^{n}$ $r = (-\frac{4}{5})$

(c) $\sum_{n=1}^{\infty} \left(\frac{\pi}{2}\right)^{n}$

(d) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ $r = \frac{1}{2}$

a.) $\frac{a_1}{1-r} = \frac{3}{1-.75} = \frac{3}{.25} = \boxed{12}$

b.) $\frac{1}{1-(-\frac{4}{5})} = \frac{1}{1+\frac{4}{5}} = \frac{1}{\frac{9}{5}} = \boxed{\frac{5}{9}}$

c.) $r = \frac{\pi}{2}$

Diverges

D.N.E.

d.) $\frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$

EXAMPLE 5 Converting a Repeating Decimal to Fraction Form

Express $0.\overline{234} = 0.234234234\ldots$ in fraction form.

$r = .001$

SOLUTION We can write this number as a sum: $0.234 + 0.000234 + 0.000000234 + \cdots$.

$$\frac{.234}{1-.001} = \frac{.234}{.999} = \frac{234}{999} = \frac{26}{111}$$

what if $3.\underline{234234}\ldots\ldots$

$$3 + \frac{26}{111} = \frac{333}{111} + \frac{26}{111} = \frac{359}{111}$$

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