

# 3

# My A, B, C, Ds

## Transformations of Exponential Functions

### Warm Up

1. Describe the effect of changing the  $A$ -value of the function  $A \cdot f(x)$ , given the basic function  $f(x) = x$ .
2. Describe the effect of changing the  $D$ -value of the function  $f(x) + D$ , given the basic function  $f(x) = x$ .

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### Learning Goals

- Graph exponential functions and transformations of exponential functions.
- Graph and analyze vertical translations and horizontal translations of exponential functions.
- Graph and analyze horizontal and vertical reflections of exponential functions.
- Write equations of transformed functions from a graph.
- Write equations of transformed functions from a description.
- Rewrite exponential functions in different forms.

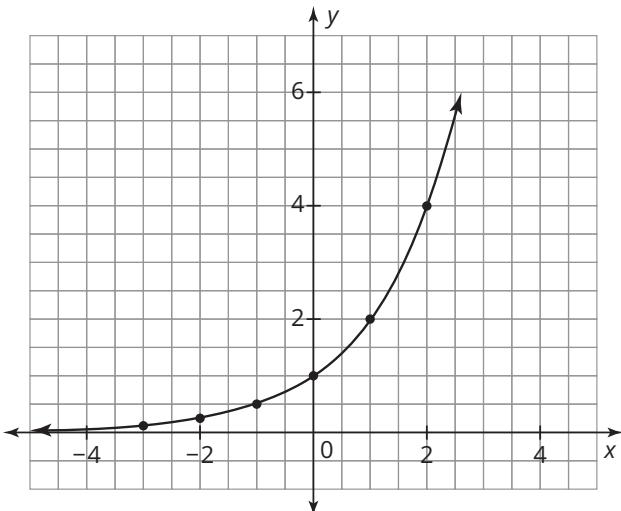
### Key Terms

- argument of a function
- reflection
- line of reflection

You know how to transform linear and absolute value functions. Do transformations of exponential functions behave in the same way?

**H, I, J, ...**

Consider the function graphed.



**1. Identify each part of the graphed function.**

a. the domain

b. the range

c. the horizontal asymptote

Recall that the transformation form of a function  $y = f(x)$ , can be written as shown.

outside the function

$$g(x) = A \cdot f(x) + D$$

**2. How do the  $A$  and  $D$  values affect the graph of  $f(x)$ ?**

## ACTIVITY

**3.1**

# Vertical Translations of Exponential Functions



Consider the three exponential functions:  $h$ ,  $s$ , and  $t$ .

$$h(x) = 2^x \quad s(x) = 2^x + 3 \quad t(x) = 2^x - 3$$

In this case,  $h(x) = 2^x$  is the basic function because it is the simplest exponential function with a base of 2. It is in the form  $f(x) = a \cdot b^x$ , where  $a = 1$  and  $b = 2$ .

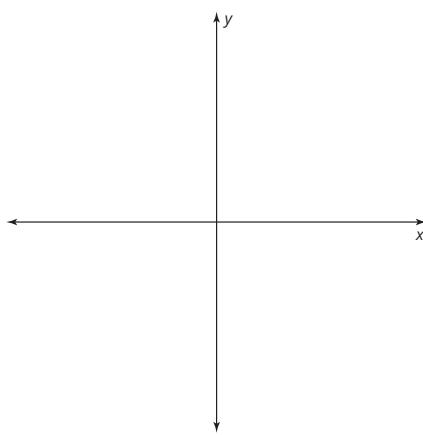
- 1. Write the functions  $s(x)$  and  $t(x)$  in terms of the basic function  $h(x)$ . Then, describe the operation performed on the basic function  $h(x)$  to result in each of the equations for  $s(x)$  and  $t(x)$ .**

$$s(x) = \underline{\hspace{2cm}}$$

$$t(x) = \underline{\hspace{2cm}}$$

- 2. Explain how you know that the graphs of  $s(x)$  and  $t(x)$  are vertical translations of the graph of  $h(x)$ .**

- 3. Sketch and label the graphs of each function. Identify key points.**



When graphing an exponential function, consider the points when  $x = -1, 0$ , and  $1$ .

- 4. Compare the  $y$ -intercepts of the graphs of  $s(x)$  and  $t(x)$  to the  $y$ -intercept of the graph of the basic function  $h(x)$ . What do you notice?**
- 5. Compare the horizontal asymptotes of the graphs of  $s(x)$  and  $t(x)$  to the horizontal asymptote of the graph of the basic function  $h(x)$ . What do you notice?**
- 6. Write the  $y$ -value of each of the corresponding reference points on  $s(x)$  and  $t(x)$ .**

$h(x) = 2^x$	$s(x) = 2^x + 3$	$t(x) = 2^x - 3$
$(-2, \frac{1}{4})$	$(-2, \underline{\hspace{2cm}})$	$(-2, \underline{\hspace{2cm}})$
$(-1, \frac{1}{2})$	$(-1, \underline{\hspace{2cm}})$	$(-1, \underline{\hspace{2cm}})$
$(0, 1)$	$(0, \underline{\hspace{2cm}})$	$(0, \underline{\hspace{2cm}})$
$(1, 2)$	$(1, \underline{\hspace{2cm}})$	$(1, \underline{\hspace{2cm}})$
$(2, 4)$	$(2, \underline{\hspace{2cm}})$	$(2, \underline{\hspace{2cm}})$

- 7. Use the table to compare the ordered pairs of the graphs of  $s(x)$  and  $t(x)$  to the ordered pairs of the graph of the basic function  $h(x)$ . What do you notice?**

**8. Complete each sentence with the coordinate notation to represent the vertical translation of each function.**

a.  $s(x) = h(x) + 3$

Each point  $(x, y)$  on the graph of  $h(x)$  becomes the point \_\_\_\_\_ on  $s(x)$ .

b.  $t(x) = h(x) - 3$

Each point  $(x, y)$  on the graph of  $h(x)$  becomes the point \_\_\_\_\_ on  $t(x)$ .

Recall that for the basic function, the  $D$ -value of the transformed function  $y = f(x) + D$  affects the output values of the function. For  $D > 0$ , the graph vertically shifts up. For  $D < 0$ , the graph vertically shifts down. The magnitude of the shift is given by  $|D|$ .

**9. What generalization can you make about the effects of vertical translations on the domain, range, and asymptotes of exponential functions?**

## ACTIVITY

**3.2**

# Horizontal Translations of Exponential Functions



The **argument of a function** is the variable on which the function operates.

Consider the three exponential functions shown, where  $h(x) = 2^x$  is the basic function. The operations are performed on  $x$ , which is the *argument of the function*.

- $h(x) = 2^x$
- $v(x) = 2^{(x + 3)}$
- $w(x) = 2^{(x - 3)}$

You can write the given functions  $v(x)$  and  $w(x)$  in terms of the basic function  $h(x)$ .

## Worked Example

To write  $v(x)$  in terms of  $h(x)$ , you just substitute  $x + 3$  into the argument for  $h(x)$ , as shown.

$$h(x) = 2^x$$

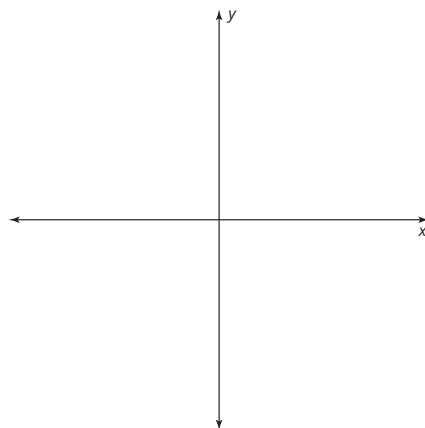
$$v(x) = h(x + 3) = 2^{(x + 3)}$$

So,  $x + 3$  replaces the variable  $x$  in the function  $h(x) = 2^x$ .

Sketch the graphs one at a time to help you see which is which.

**1. Write the function  $w(x)$  in terms of the basic function  $h(x)$ .**

**2. Sketch and label the graph of each function. Identify key points.**



- 3. Compare the graphs of  $v(x)$  and  $w(x)$  to the graph of the basic function. What do you notice?**

- 4. Write the  $x$ -value of each of the corresponding reference points on  $v(x)$  and  $w(x)$ .**

$h(x) = 2^x$	$v(x) = 2^{(x+3)}$	$w(x) = 2^{(x-3)}$
$(-2, \frac{1}{4})$	$(\underline{\hspace{2cm}}, \frac{1}{4})$	$(\underline{\hspace{2cm}}, \frac{1}{4})$
$(-1, \frac{1}{2})$	$(\underline{\hspace{2cm}}, \frac{1}{2})$	$(\underline{\hspace{2cm}}, \frac{1}{2})$
$(0, 1)$	$(\underline{\hspace{2cm}}, 1)$	$(\underline{\hspace{2cm}}, 1)$
$(1, 2)$	$(\underline{\hspace{2cm}}, 2)$	$(\underline{\hspace{2cm}}, 2)$
$(2, 4)$	$(\underline{\hspace{2cm}}, 4)$	$(\underline{\hspace{2cm}}, 4)$

- 5. Use the table to compare the ordered pairs of the graphs of  $v(x)$  and  $w(x)$  to the ordered pairs of the graph of the basic function  $h(x)$ . What do you notice?**



Notice there are no negative  $y$ -values in this table. Are negative values included in the range of  $h$ ,  $v$ , or  $w$ ?

If a constant is added or subtracted outside a function, like  $g(x) + 3$  or  $g(x) - 3$ , then only the  $y$ -values change, resulting in a vertical translation.

And, if a constant is added or subtracted inside a function, like  $g(x + 3)$  or  $g(x - 3)$ , then only the  $x$ -values change, resulting in a horizontal translation.

**6. Complete each sentence with the coordinate notation to represent the horizontal translation of each function.**

a.  $v(x) = h(x + 3)$

Each point  $(x, y)$  on the graph of  $h(x)$  becomes the point \_\_\_\_\_ on  $v(x)$ .

b.  $w(x) = h(x - 3)$

Each point  $(x, y)$  on the graph of  $h(x)$  becomes the point \_\_\_\_\_ on  $w(x)$ .

**7. Describe each graph in relation to the basic function  $h(x) = b^x$ .**

a. Compare  $f(x) = h(x - C)$  to the basic function for  $C > 0$ .

b. Compare  $f(x) = h(x - C)$  to the basic function for  $C < 0$ .

For the basic function, the  $C$ -value of the transformed function  $y = f(x - C)$  affects the input values of the function. The value  $|C|$  describes the number of units the graph of  $f(x)$  is translated right or left. If  $C > 0$ , the graph is translated to the right. If  $C < 0$ , the graph is translated to the left.

**8. What generalization can you make about the effects of horizontal translations on the domain, range, and asymptotes of exponential functions?**

ACTIVITY  
**3.3**

## Reflections of Exponential Functions



Consider the three exponential functions shown, where  $h(x) = 2^x$  is the basic function.

- $h(x) = 2^x$
- $m(x) = -(2^x)$
- $n(x) = 2^{(-x)}$

**1. Write the functions  $m(x)$  and  $n(x)$  in terms of the basic function  $h(x)$ .**

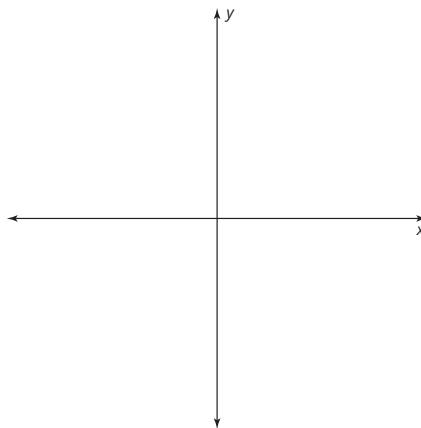
$m(x) =$  \_\_\_\_\_

$n(x) =$  \_\_\_\_\_

**2. Compare  $m(x)$  to  $h(x)$ . Does an operation performed on  $h(x)$  or on the argument of  $h(x)$  result in the equation for  $m(x)$ ? What is the operation?**

**3. Compare  $n(x)$  to  $h(x)$ . Does an operation performed on  $h(x)$  or on the argument of  $h(x)$  result in the equation for  $n(x)$ ? What is the operation?**

**4. Use technology to sketch and label each function.**



**5. Compare the graphs of  $m(x)$  and  $n(x)$  to the graph of the basic function  $h(x)$ . What do you notice?**

**6. Write the  $y$ -value of each of the corresponding reference points on  $m(x)$  and  $n(x)$ .**

$h(x) = 2^x$	$m(x) = -(2^x)$	$n(x) = 2^{(-x)}$
$(-2, \frac{1}{4})$	$(-2, \underline{\hspace{1cm}})$	$(\underline{\hspace{1cm}}, \frac{1}{4})$
$(-1, \frac{1}{2})$	$(-1, \underline{\hspace{1cm}})$	$(\underline{\hspace{1cm}}, \frac{1}{2})$
$(0, 1)$	$(0, \underline{\hspace{1cm}})$	$(\underline{\hspace{1cm}}, 1)$
$(1, 2)$	$(1, \underline{\hspace{1cm}})$	$(\underline{\hspace{1cm}}, 2)$
$(2, 4)$	$(2, \underline{\hspace{1cm}})$	$(\underline{\hspace{1cm}}, 4)$

**7. Use the table to compare the ordered pairs of the graphs of  $m(x)$  and  $n(x)$  to the ordered pairs of the graph of the basic function  $h(x)$ . What do you notice?**

When the negative is on the outside of the function, like  $-g(x)$ , all the  $y$ -values become the opposite of the  $y$ -values of  $g(x)$ . The  $x$ -values remain unchanged.

A **reflection** of a graph is a mirror image of the graph about a *line of reflection*. A **line of reflection** is the line that the graph is reflected across. A horizontal line of reflection affects the  $y$ -coordinates, and a vertical line of reflection affects the  $x$ -coordinates.

**8. Consider the graphs of  $m(x)$  and  $n(x)$ .**

**a. Which function represents a reflection of  $h(x)$  across a horizontal line? Name the line of reflection.**

**b. Which function represents a reflection of  $h(x)$  across a vertical line? Name the line of reflection.**

When the negative is on the inside of the function, like  $g(-x)$ , all the  $x$ -values become the opposite of the  $x$ -values of  $g(x)$ . The  $y$ -values remain unchanged.

**9. Complete each sentence with the coordinate notation to represent the reflection of each function.**

**a.  $m(x) = -h(x)$**

**Each point  $(x, y)$  on the graph of  $h(x)$  becomes the point \_\_\_\_\_ on  $m(x)$ .**

**b.  $n(x) = h(-x)$**

**Each point  $(x, y)$  on the graph of  $h(x)$  becomes the point \_\_\_\_\_ on  $n(x)$ .**

Recall that for the basic function, the  $A$ -value of the transformed function  $y = A \cdot f(x)$  affects the output values of the function. For  $|A| > 1$ , the graph vertically stretches. For  $0 < |A| < 1$ , the graph vertically compresses. For  $A = -1$ , the graph is reflected across the line  $y = 0$ , or the  $x$ -axis.

In this activity, you considered a different transformation that affects the input values of a function. For the basic function, the  $B$ -value of the transformed function  $y = f(Bx)$  affects the input values of the function. For  $B = -1$ , the graph is reflected across the line  $x = 0$ , or the  $y$ -axis.

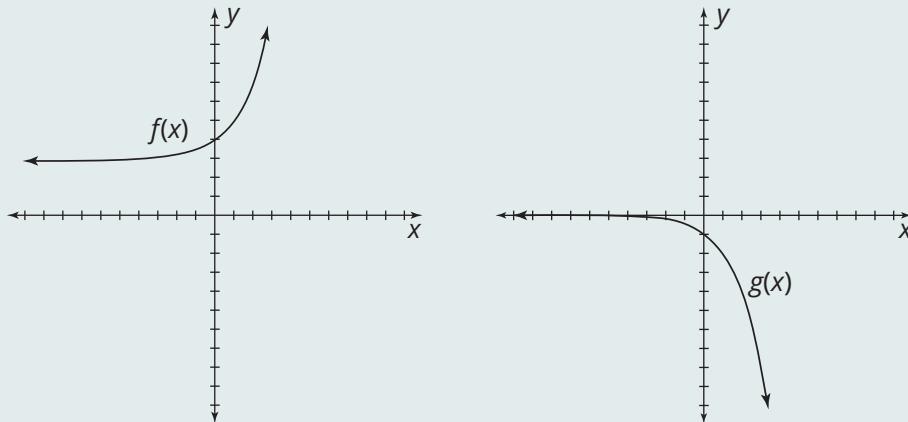
## ACTIVITY

**3.4**

# Interpreting and Graphing Exponential Functions



There are different ways to interpret equations of exponential functions and transformations of exponential functions.

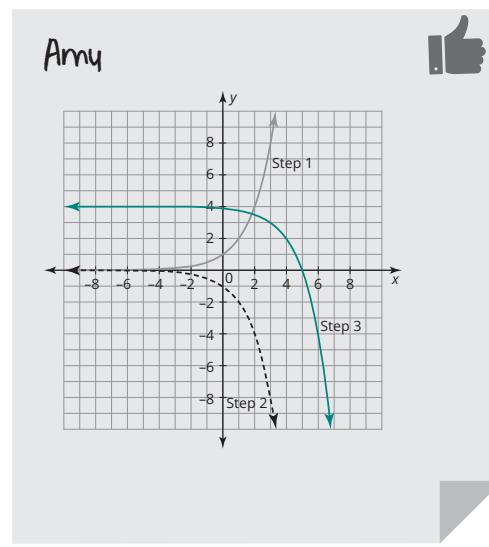
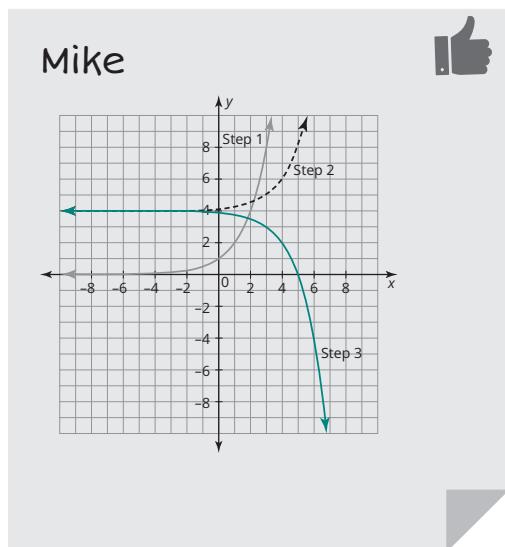
**1. Jacob and Kate are comparing the two graphs shown.**

Jacob says that to get the graph of  $g(x)$ , first translate  $f(x)$  down 3 units, and then reflect across the line  $y = 0$ . Kate says that to get the graph of  $f(x)$ , first reflect  $g(x)$  across the line  $y = 0$ , and then translate up 3 units. Who is correct? Explain your reasoning.

You know that changing the  $A$ -value of a function to its opposite reflects the function across a horizontal line. But the line of reflection for the function might be different depending on how you write the transformation and the order the transformations are applied.

**2. Consider the function  $f(x) = -2^{(x-3)} + 4$ .**

- a. Mike and Amy used the basic function in different ways to graph  $f(x)$ . Provide the step-by-step reasoning used by each student.**

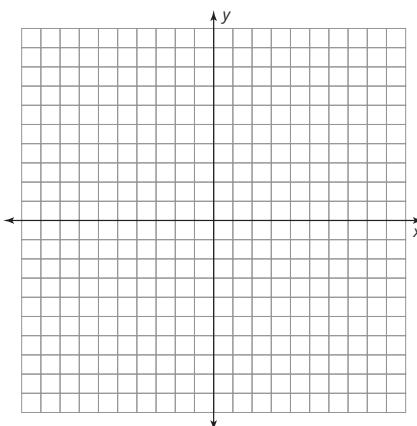


- b. Explain how changing the order of the transformations affects the line of reflection.**

**3. Use the given characteristics to write an equation and then graph  $f'(x)$ , given the basic function  $f(x) = 2^x$ . Label key points.**

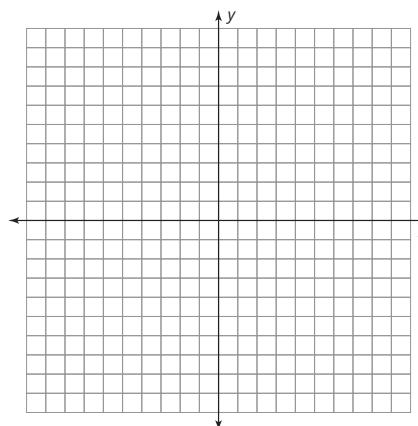
**a.  $f'(x) = f(x) + 5$**

**Equation:  $f'(x) =$  \_\_\_\_\_**



**b.  $f'(x) = -f(x) + 5$**

**Equation:  $f'(x) =$  \_\_\_\_\_**

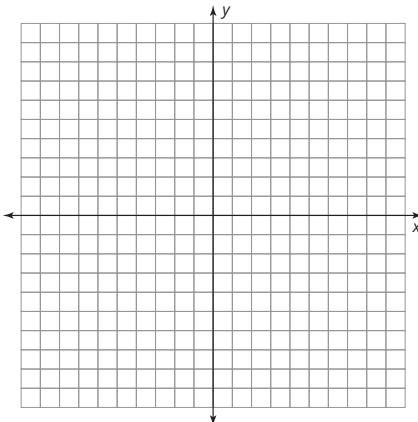


One way to indicate the transformation of a function is by using the prime symbol.

The function  $f'(x)$  is a transformation of  $f(x)$ .

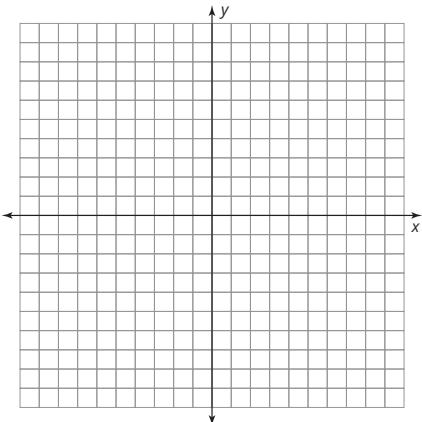
c.  $f'(x) = f(-x) + 5$

Equation:  $f'(x) =$  \_\_\_\_\_



d.  $f'(x) = -f(x) - 5$

Equation:  $f'(x) =$  \_\_\_\_\_



You have analyzed different ways to graph transformations of the basic function  $h(x) = 2^x$ . Now, let's consider different ways to interpret the equations of function transformations.

4. Andres and Tomas each described the effects of transforming the graph of  $f(x) = 3^x$ , such that  $p(x) = 3f(x)$ . Who's correct? Explain your reasoning.



Andres

$$p(x) = 3f(x)$$

The A-value is 3 so the graph is stretched vertically by a scale factor of 3.

Tomas

$$p(x) = 3f(x)$$

$$p(x) = 3 \cdot 3^x$$

$$p(x) = 3^{(1+x)}$$

$$p(x) = f(x+1)$$

The C-value is -1 so the graph is horizontally translated 1 unit to the left.

- 5. Devonte says that you can rewrite the equation for  $n(x) = 2^{(-x)}$  with a  $b$ -value equal to  $\frac{1}{2}$ .**



**Is Devonte correct? Explain why or why not.**

An exponential function can be rewritten to show an expression with no  $C$ -value transformation.

### Worked Example

Given the function  $h(x) = 2^x$ , consider the function  $v(x) = h(x + 3)$ .

$$v(x) = h(x + 3)$$

$$v(x) = 2^{x+3}$$

You can rewrite  $v(x)$  with no  $C$ -value.

$$v(x) = 2^{x+3}$$

$$= 2^x \cdot 2^3$$

$$= 8 \cdot 2^x$$

### 6. Given the function $f(x) = 2^x$ .

**a. Rewrite  $c(x) = f(x - 2)$  as an exponential function with no  $C$ -value transformation.**

**b. Rewrite  $b(x) = f(x + 2)$  as an exponential function with no  $C$ -value transformation.**

ACTIVITY  
**3.5**

## Writing Exponential Functions Given Graphs and Descriptions

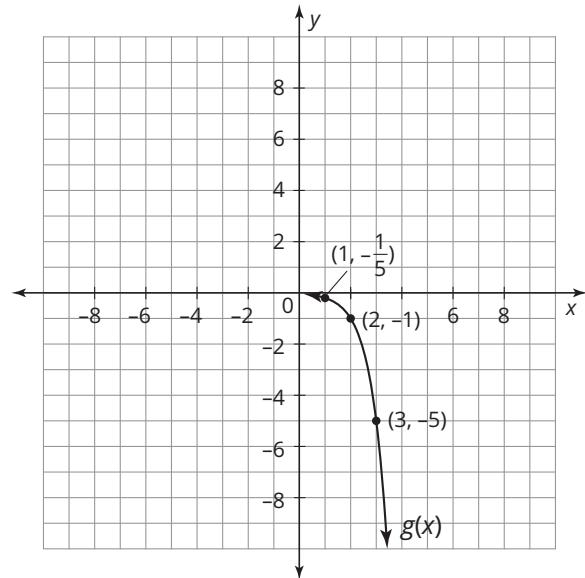
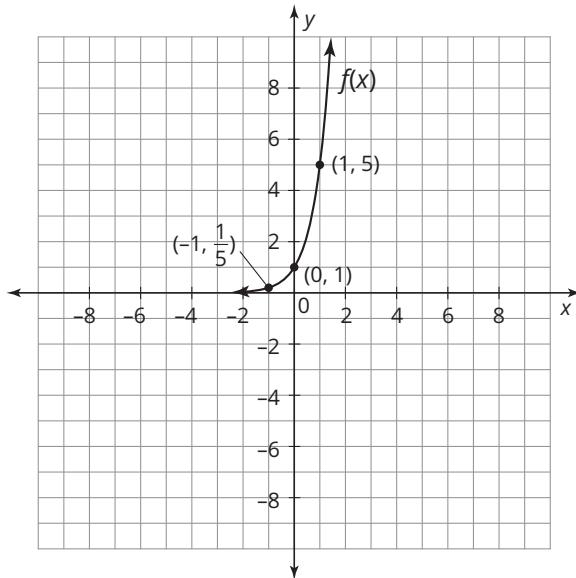


1. Consider the function,  $f(x) = 2^x$ . Write the function in transformation form in terms of the transformations described, then write an equivalent equation.

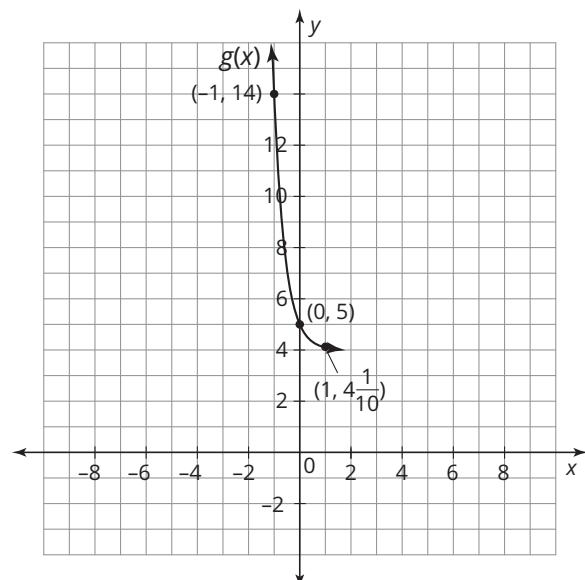
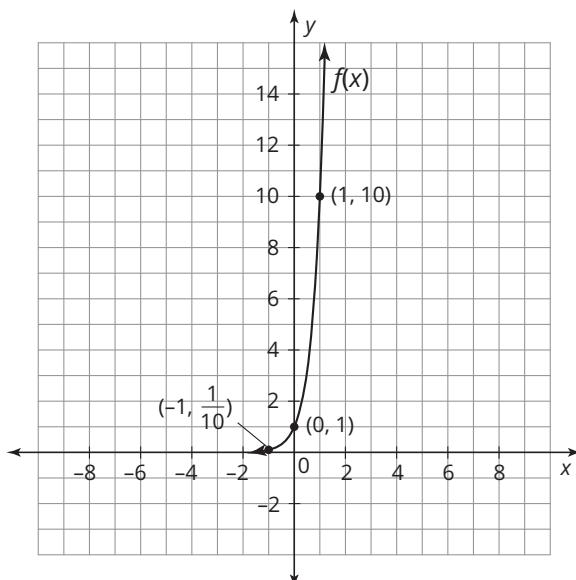
Transformation	Transformation Function Form	Equation
a. Reflection across the $y$ -axis		
b. Reflection across the $x$ -axis		
c. Horizontal translation of 2 units to the left and a vertical translation of 3 units up		
d. Vertical stretch of 2 units and a reflection across the line $y = 0$		
e. Reflection across the line $y = 3$		
f. Horizontal translation of 3 units to the right, a vertical translation down 2 units, and a vertical dilation of $\frac{1}{2}$		
g. Vertical compression by a factor of 4		
h. Vertical stretch by a factor of 4		

- 2. Analyze the graphs of  $f(x)$  and  $g(x)$ . Describe the transformations performed on  $f(x)$  to create  $g(x)$ . Then, write an equation for  $g(x)$  in terms of  $f(x)$ . For each set of points shown on  $f(x)$ , the corresponding points are shown on  $g(x)$ .**

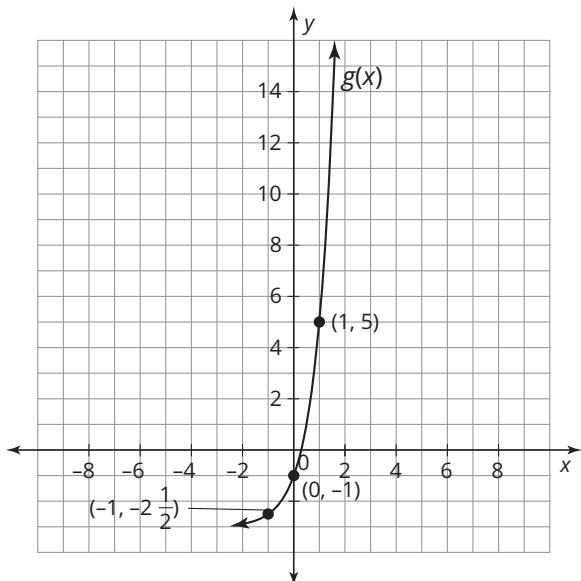
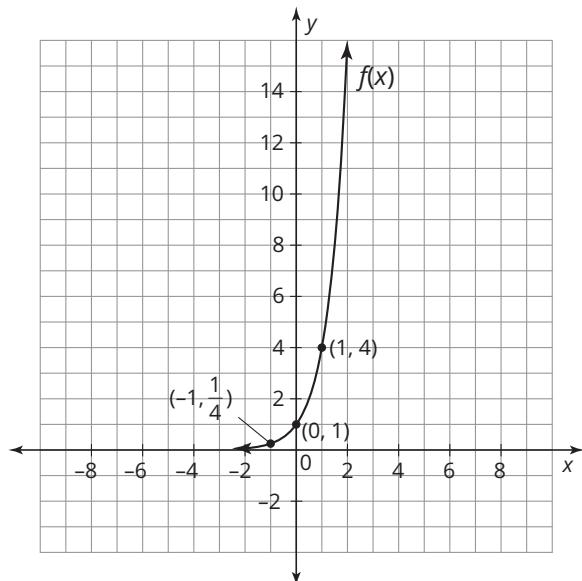
a.  $g(x) = \underline{\hspace{2cm}}$



b.  $g(x) = \underline{\hspace{2cm}}$



c.  $g(x) = \underline{\hspace{2cm}}$



## TALK the TALK



### Next Time, Won't You Sing With Me?

1. Determine whether each statement is true or false. If the statement is false, rewrite the statement as true.
  - a. In the transformation form of  $f(x)$ ,  $g(x) = Af(x - C) + D$ , the  $D$ -value translates the function  $f(x)$  horizontally, the  $C$ -value translates  $f(x)$  vertically, and the  $A$ -value horizontally stretches or compresses  $f(x)$ .
  - b. Key characteristics of basic exponential functions include a domain of nonnegative numbers, a range of real numbers, and a vertical asymptote at  $y = 0$ .
  - c. The domain of exponential functions is not affected by translations or dilations.
  - d. Vertical translations do not affect the range and the horizontal asymptote of exponential functions.
  - e. Horizontal translations do not affect the range and the horizontal asymptote of exponential functions.
  - f. Vertical dilations do not affect the range and the horizontal asymptote of exponential functions.
2. The basic exponential function can be written as  $f(x) = A \cdot 2^{(x - C)} + D$ . When  $A = 1$ ,  $C = 0$ , and  $D = 0$ , then the function is equivalent to  $f(x) = 2^x$ . Complete the graphic organizer to summarize the transformations of an exponential function.

# Graphic Organizer

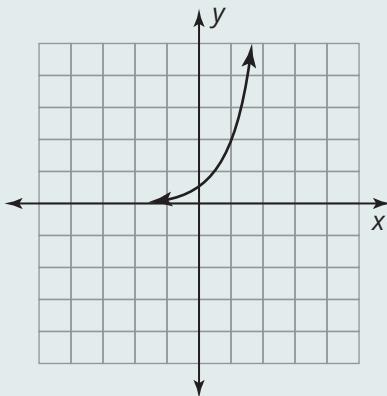
## A-Value

$$y = A \cdot 2^x$$

$$|A| > 1$$

$$0 < |A| < 1$$

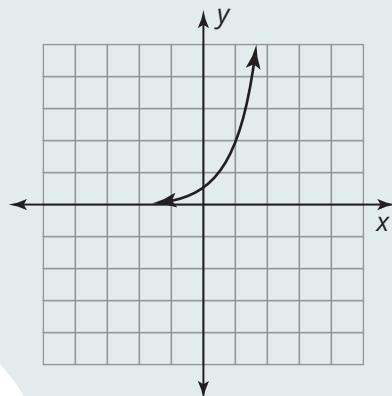
$$A = -1$$



## B-Value

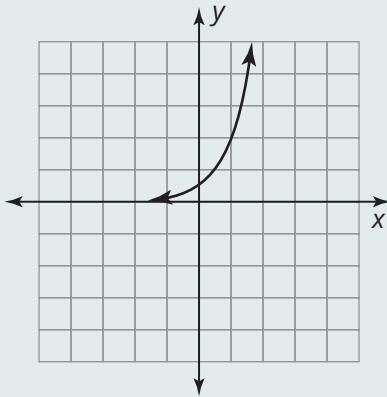
$$y = 2^{Bx}$$

$$B = -1$$



$$y = 2^x$$

y



$$C > 0$$

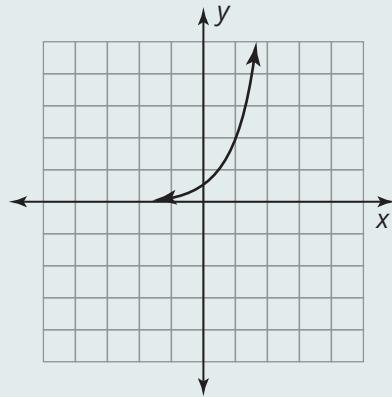
$$C < 0$$

$$y = 2^{x-C}$$

## C-Value

y

y



$$D > 0$$

$$D < 0$$

$$y = 2^x + D$$

## D-Value

# Assignment

## Write

Given a basic function and the equation for a reflection of a basic function, explain how to determine whether the line of reflection will be the  $x$ -axis or the  $y$ -axis.

## Remember

Transformations performed on any function  $f(x)$  can be described by the transformation function  $g(x) = Af(x - C) + D$  where the  $D$ -value translates the function  $f(x)$  vertically, the  $C$ -value translates  $f(x)$  horizontally, and the  $A$ -value vertically stretches or compresses  $f(x)$ .

## Practice

1. Complete the table to determine the corresponding points on  $c(x)$ , given reference points on  $f(x)$ . Then, graph  $c(x)$  on the same coordinate plane as  $f(x)$  and state the domain, range, and asymptotes of  $c(x)$ .

a.  $f(x) = 2^x$

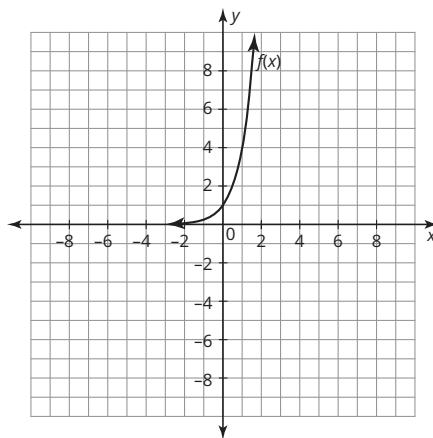
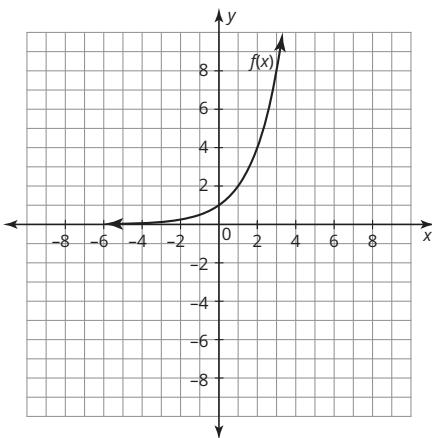
$c(x) = f(x - 1)$

Reference Points on $f(x)$	Corresponding Points on $c(x)$
$(-1, \frac{1}{2})$	
$(0, 1)$	
$(1, 2)$	

b.  $f(x) = 4^x$

$c(x) = -f(x) - 2$

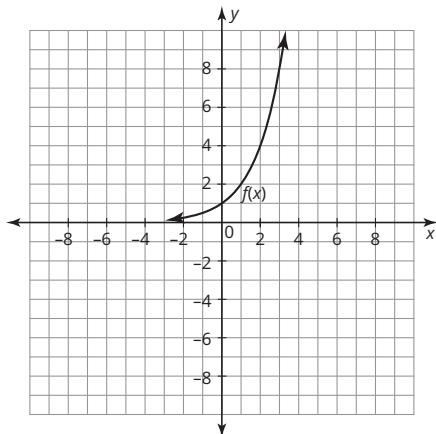
Reference Points on $f(x)$	Corresponding Points on $c(x)$
$(-1, \frac{1}{4})$	
$(0, 1)$	
$(1, 4)$	



c.  $f(x) = 2^x$

$c(x) = 4f(x)$

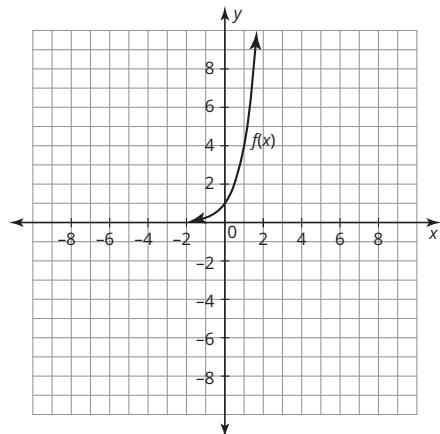
Reference Points on $f(x)$	Corresponding Points on $c(x)$
$(-1, \frac{1}{2})$	
$(0, 1)$	
$(1, 2)$	



d.  $f(x) = 4^x$

$c(x) = f(-x)$

Reference Points on $f(x)$	Corresponding Points on $c(x)$
$(-1, \frac{1}{4})$	
$(0, 1)$	
$(1, 4)$	



2. Describe the transformations performed on  $m(x)$  that produced  $t(x)$ . Then, write an exponential equation for  $t(x)$ .

a.  $m(x) = 3^x$

$t(x) = -m(x + 1)$

b.  $m(x) = 5^x$

$t(x) = 3m(x) - 2$

c.  $m(x) = 4^x$

$t(x) = m(x - 1)$

d.  $m(x) = 7^x$

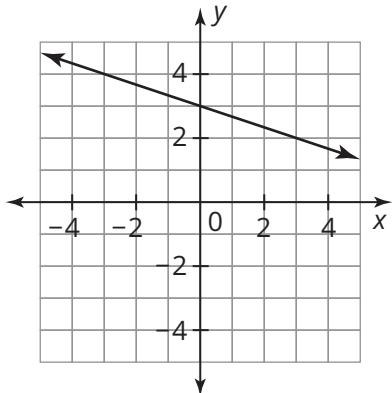
$t(x) = m(x) + 2$

## Stretch

Research real-world examples for which exponential functions provide good models. Write a short paragraph explaining why an exponential model works well for at least one of the examples.

## Review

1. Given  $f(x) = x$ , write  $m(x)$  in terms of a transformation of  $f(x)$ .



2. Solve each equation.

a.  $2^{x+1} = 64$

b.  $4^x = 64$

3. Determine the location of a point labeled  $G$  on the coordinate plane such that the figure  $DEFG$  is a square. Justify your reasoning.

