Downtown and Uptown

Exponential Equations for Growth and Decay

Warm Up

Determine the constant ratio for each geometric sequence.

1. 10, 10.5, 11.025, 11.57625, 12.1550625 . . .

2. 27, 9, 3, 1, $\frac{1}{3}$ . . .

3. 1, $\frac{3}{4}$, $\frac{9}{16}$, $\frac{27}{64}$ . . .

Learning Goals

- Classify exponential functions as increasing or decreasing.
- Compare formulas for simple interest and compound interest situations.
- Compare the average rate of change between common intervals of a linear and an exponential relationship.
- Write an exponential function that includes a percent increase or decrease with a b-value that is a decimal number.
- Solve exponential equations using graphs.

Key Terms

- simple interest
- compound interest
- exponential growth function
- exponential decay function

You have analyzed linear and exponential functions and their graphs. How can you compare linear and exponential functions as increasing and decreasing functions?

Up or Down?

Consider each function shown.

$$f(x) = -2x + 5 \qquad g(x) = 2^x - 1 \qquad h(x) = 0.95^x$$

$$p(x) = 6 \cdot \left(\frac{5}{8}\right)^x + 2 \qquad q(x) = 3(x - 4) - 1 \qquad r(x) = 2 \cdot (1 - 0.5)^x$$

$$v(x) = 4 \cdot 1.10^{(x + 5)} \qquad w(x) = -5 \cdot 3^x + 1 \qquad z(x) = -x + 10$$

1. Sort the functions into two groups. Justify your choices.

Increasing Functions	Decreasing Functions

ACTIVITY 1.1

Simple and Compound Interest

Suppose that your family deposited $10,000 in an interest bearing account for your college fund that earns simple interest each year. A friend's family deposited $10,000 in an interest bearing account for their child's college fund that earns compound interest each year.
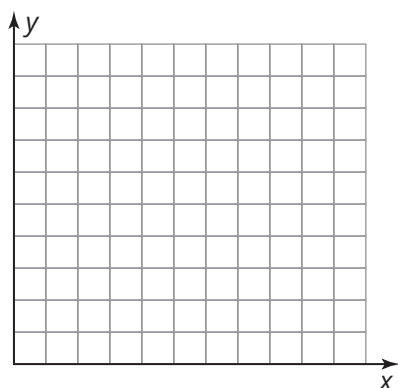
Time (years)	Simple Interest Balance (dollars)	Compound Interest Balance (dollars)
0	10,000	10,000
1	10,400	10,400
2	10,800	10,816
3	11,200	11,248.64
10	14,000	14,802.44

In a **simple interest** account, a percent of the starting balance is added to the account at each interval. The formula for simple interest is $I = Prt$, where P represents the starting amount, or principal, r represents the interest rate, t represents time, and I represents the interest earned. In a **compound interest** account, the balance is multiplied by the same amount at each interval.
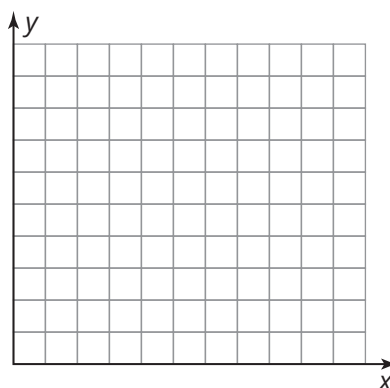
1. Study the table of values.

a. Sketch a graph of each account balance in dollars as a function of the time in years.

Simple Interest Balance

Compound Interest Balance

b. Write a function, $s(x)$, to represent the simple interest account and a function, $c(x)$, to represent the compound interest account.

2. Use the functions $s(x)$ and $c(x)$ to determine each value.

   a. $s(5)$ b. $c(5)$

   c. $c(4)$ d. $s(4)$

3. Determine the average rate of change between each pair of values given for each relationship.

Time Intervals (years)	Simple Interest Function (dollars)	Compound Interest Function (dollars)
Between $t = 0$ and $t = 1$		
Between $t = 1$ and $t = 2$		
Between $t = 2$ and $t = 5$		
Between $t = 5$ and $t = 10$		

4. Compare the average rates of change for the simple and compound interest accounts.

   a. What do you notice?

   b. What does this tell you about the graphs of linear and exponential functions?

5. **Use technology to determine when each account will reach the given dollar amount.**

   a. **When does the simple interest account reach $15,600?**

   b. **Approximately when does the compound interest account reach one million dollars?**

6. **Chloe says that given any increasing linear function and any exponential growth function, the output of the exponential function will eventually be greater than the output of the linear function. Is Chloe correct? Use examples to justify your thinking.**
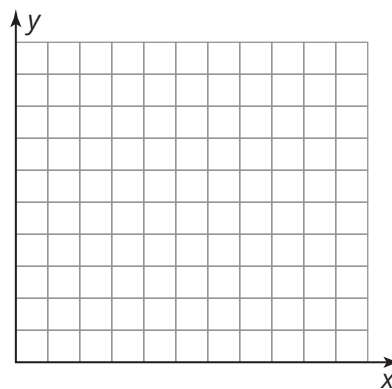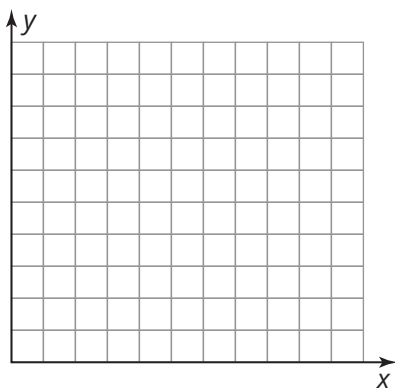
Identifying Exponential Growth and Decay

At this moment, the population of Downtown is 20,000 people, and the population of Uptown is 6000 people. But over many years, people have been moving away from Downtown at a rate of 1.5% every year. At the same time, Uptown's population has been growing at a rate of 1.8% each year.

1. **What are the independent and dependent quantities in each situation?**

2. **Which city's population can be represented as an increasing function, and which can be represented as a decreasing function?**

Let's examine the properties of the graphs of the functions for Downtown and Uptown.

Downtown: $D(t) = 20{,}000(1 - 0.015)^t$ Uptown: $U(t) = 6000(1 + 0.018)^t$

3. **Sketch a graph of each function. Label key points.**

4. The functions $D(t)$ and $U(t)$ can each be written as an exponential function of the form $f(x) = a \cdot b^x$.

a. What is the a-value for each function? What does each a-value mean in terms of this problem situation?

b. What is the b-value for each function? What does each b-value mean in terms of this problem situation?

c. Compare and explain the meanings of the expressions $(1 - 0.015)^t$ and $(1 + 0.018)^t$ in terms of this problem situation.

5. Analyze the y-intercepts of each function.

a. Identify the y-intercepts.

b. Interpret the meaning of the y-intercept in terms of the problem situation.

c. Describe how you can determine the y-intercept of each function using only the formula for population increase or decrease.

Think about:

A decreasing exponential function is denoted by a decimal or fractional b-value between 0 and 1, not by a negative b-value.

An **exponential growth function** has a b-value greater than 1 and is of the form $y = a \cdot (1 + r)^x$, where r is the rate of growth. The b-value is $1 + r$. An **exponential decay function** has a b-value greater than 0 and less than 1 and is of the form $y = a \cdot (1 - r)^x$, where r is the rate of decay. The b-value is $1 - r$.

TALK the TALK 💬

Imagine All the People . . .

Consider the six different population scenarios.

1. Match each situation with the appropriate function. Explain your reasoning.

Functions

$f(x) = 7000 \cdot 0.969^x$

$f(x) = 7000 \cdot (1 + 0.028)^x$

$f(x) = 7000 \cdot 1.012^x$

$f(x) = 7000 \cdot (1 - 0.0175)^x$

$f(x) = 7000 \cdot 1.014^x$

$f(x) = 7000 \cdot 0.9875^x$

Blueville has a population of 7000. Its population is increasing at a rate of 1.4%.

Greenville has a population of 7000. Its population is decreasing at a rate of 1.75%.

Youngstown has a population of 7000. Its population is increasing at a rate of 1.2%.

North Park has a population of 7000. Its population is decreasing at a rate of 3.1%.

West Lake has a population of 7000. Its population is increasing at a rate of 2.8%.

Springfield has a population of 7000. Its population is decreasing at a rate of 1.25%.

Write

Explain the difference between simple interest and compound interest.

Remember

An exponential growth function has a *b*-value greater than 1 and is of the form $y = a \cdot (1 + r)^x$, where *r* is the rate of growth. An exponential decay function has a *b*-value greater than 0 and less than 1 and is of the form $y = a \cdot (1 - r)^x$, where *r* is the rate of decay.

Practice

1. Chanise just received a $2500 bonus check from her employer. She is going to put it into an account that will earn interest. The Basic savings account at her bank earns 6% simple interest. The Gold savings account earns 4.5% compound interest.
   a. Write a function for each account that can be used to determine the balance in the account based on the year, *t*. Describe each function.
   b. Use your answers to part (a) to create a table of values for each function.
   c. Use technology to graph the functions for the Basic and Gold savings accounts. Then, sketch the graphs.
   d. Into which account would you recommend that Chanise deposit her money? Explain your reasoning.
   e. After reading the pamphlet about the different accounts a little more closely, Chanise realizes that there is a one-time fee of $300 for depositing her money in the Gold account. Does this change the recommendation you made in part (d)? Why or why not?
   f. Compare the rates of change for the Basic and Gold savings accounts. Explain what the rates of change tell you about the accounts.
   g. What do the rates of change for linear and exponential functions tell you about the graphs of the functions?
2. Ainsley works for the owners of a bookstore. Her starting salary was $24,500, and she gets a 3% raise each year.
   a. Write an equation in function notation to represent Ainsley's salary as a function of the number of years she has been working at the bookstore.
   b. What will Ainsley's salary be when she begins her fourth year working at the bookstore? Show your work.

Stretch

Consider a piece of paper that is 0.1 mm thick. How many times must it be folded so that it reaches the top of the Eiffel Tower? Assume the paper is as large as needed, and it is possible to fold it as many times as required.

Review

1. Roberto and Maeko open a pet store and start with 5 hamsters for sale. Hamster populations usually triple every cycle. One cycle is equal to 4 months. Write an equation in function notation to represent the change in the number of hamsters as a function of the cycle number, c. Explain how you determined your equation.

2. Write an exponential function to model this table of values.

x	$g(x)$
1	0.6
2	0.06
3	0.006
4	0.0006

3. Write a function, $g(x)$, and sketch a graph that is translated 3 units up from and 4 units to the right of $f(x) = \left(\frac{1}{2}\right)^x$.

4. Write a function, $h(x)$, and sketch a graph that is a reflection of $f(x) = -3^x$ across the line $y = 0$.

5. Solve each equation for x and justify each step.
   a. $-3x = -18$
   b. $\frac{(x + 4)}{2} = \frac{(x - 5)}{3}$