Dot Product of Vectors

DEFINITION Dot Product

The **dot product** or **inner product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
3. $\mathbf{0} \cdot \mathbf{u} = 0$
4. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

   $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
5. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$

EXAMPLE 1 Finding Dot Products

Find each dot product.

(a) $\langle 3, 4\rangle \cdot \langle 5, 2\rangle$

$3 \cdot 5 + 2 \cdot 4$

$15 + 8 = \boxed{23}$

(b) $\langle 1, -2\rangle \cdot \langle -4, 3\rangle$

$1(-4) + (-2)(3)$

$-4 + (-6) = \boxed{-10}$

(c) $(2\mathbf{i} - \mathbf{j}) \cdot (3\mathbf{i} - 5\mathbf{j})$

$x \quad y \quad x \quad y$

$2(3) + (-1)(-5)$

$6 + 5 = \boxed{11}$

EXAMPLE 2 Using Dot Product to Find Length

Use the dot product to find the length of the vector $\mathbf{u} = \langle 4, -3 \rangle$.

$u = \langle 4, -3 \rangle$

$u \cdot u = |u|^2$

$16 + 9 = |u|^2$

$25 = |u|^2$

$|u| = \sqrt{25} = \boxed{5}$

$\sqrt{a^2 + b^2}$

$\sqrt{16 + 9}$

Same thing!

EXAMPLE 3 Finding the Angle Between Vectors

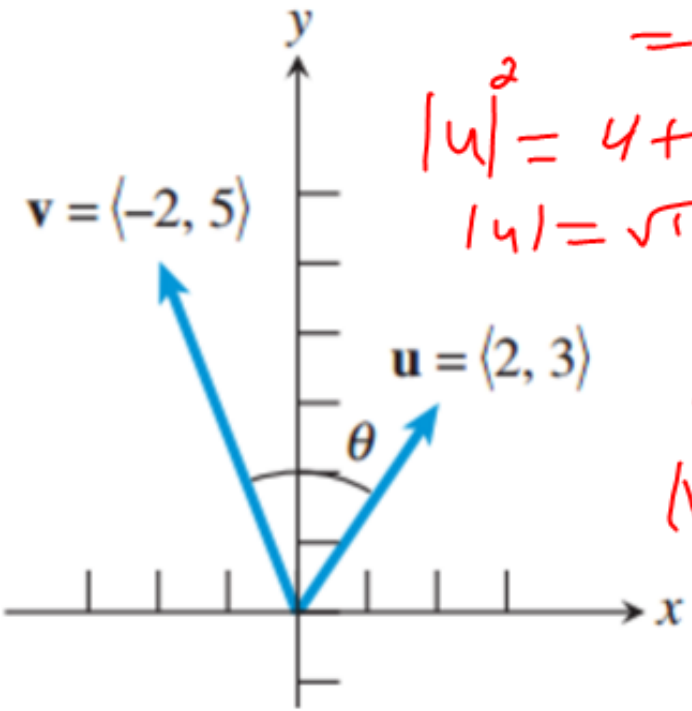
Find the angle between the vectors **u** and **v**.

(a) $\mathbf{u} = \langle 2, 3\rangle$, $\mathbf{v} = \langle -2, 5\rangle$

$u \cdot v = -4 + 15$
$= 11$

$|u|^2 = 4 + 9$
$|u| = \sqrt{13}$

$|v|^2 = 4 + 25$
$|v| = \sqrt{29}$



$$\theta = \cos^{-1}\left[\frac{u \cdot v}{|u| \cdot |v|}\right]$$

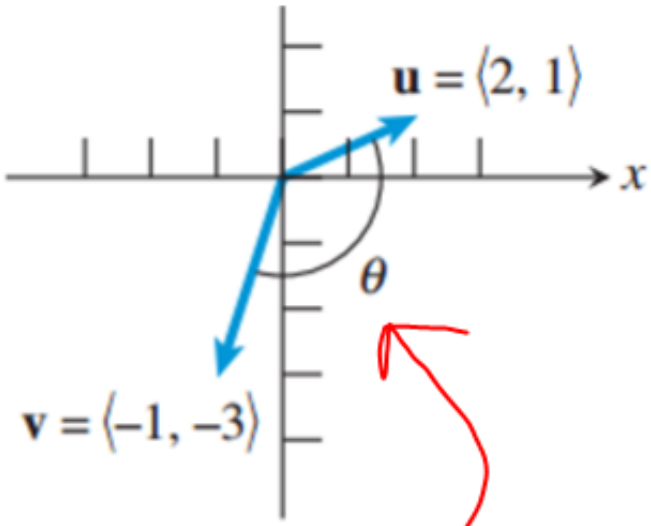
$$\theta = \cos^{-1}\left[\frac{11}{(\sqrt{13} \cdot \sqrt{29})}\right]$$

$$\theta = 55.5^\circ$$

EXAMPLE 3 Finding the Angle Between Vectors

Find the angle between the vectors **u** and **v**.

(b) $\mathbf{u} = \langle 2, 1 \rangle$, $\mathbf{v} = \langle -1, -3 \rangle$



$u \cdot v = -5$

$|u| = \sqrt{5}$

$|v| = \sqrt{10}$

$\theta = \cos^{-1}\left(\frac{-5}{\sqrt{5} \cdot \sqrt{10}}\right)$

$\theta = \cos^{-1}\left(\frac{-5}{\sqrt{50}}\right) = 135^\circ$

DEFINITION Orthogonal Vectors

The vectors **u** and **v** are **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

perpendicular

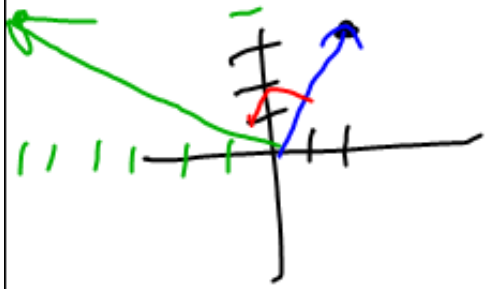
$$\frac{\cos\theta}{1} = \frac{u \cdot v}{|u| \cdot |v|}$$

$$u \cdot v = |u| \cdot |v| \cos\theta$$

$$= 0 \qquad \cos 90^\circ$$

EXAMPLE 4 Proving Vectors are Orthogonal

Prove that the vectors $\mathbf{u} = \langle 2, 3 \rangle$ and $\mathbf{v} = \langle -6, 4 \rangle$ are orthogonal.



$$u \cdot v = -12 + 12 = 0$$

$$u \cdot v = 0$$