Defining a Limit Informally

There is nothing difficult about the following limit statements:

$$\lim_{x \to 3} (2x - 1) = 5 \qquad \lim_{x \to \infty} (x^2 + 3) = \infty \qquad \lim_{n \to \infty} \frac{1}{n} = 0$$

$2(3) - 1 = 5$

DEFINITION (INFORMAL) Limit at _a_

When we write "$\lim_{x \to a} f(x) = L$," we mean that $f(x)$ gets arbitrarily close to L as x gets arbitrarily close (but not equal) to a.
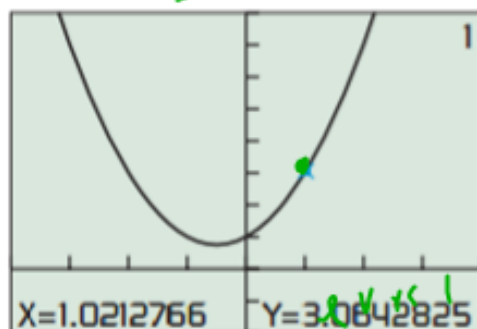
EXAMPLE 1 Finding a Limit

Find $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}. = 3$

$\dfrac{1-1}{0}$

Solve Graphically

X=1.0212766   Y=3.0842825

[−4, 4] by [−2, 8]

Solve Numerically

X	Y1	
.997	2.991	
.998	2.994	
.999	2.997	
1	ERROR	← 3
1.001	3.003	
1.002	3.006	← 3
1.003	3.009	

Y1 ▤ (X³−1)/(X−1)

(b)

Solve Algebraically

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1}$$

$(1)^2 + 1 + 1$

3

Properties of Limits

If $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ both exist, then

1. Sum Rule
$$\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

2. Difference Rule
$$\lim_{x \to c} (f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

3. Product Rule
$$\lim_{x \to c} (f(x) \cdot g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$$

4. Constant Multiple Rule
$$\lim_{x \to c} (k \cdot g(x)) = k \cdot \lim_{x \to c} g(x)$$

5. Quotient Rule
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$$

provided $\lim\limits_{x \to c} g(x) \neq 0$

6. Power Rule
$$\lim_{x \to c} (f(x))^n = (\lim_{x \to c} f(x))^n \text{ for } n$$

a positive integer

7. Root Rule
$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} \text{ for } n \geq 2$$

a positive integer, provided $\sqrt[n]{\lim\limits_{x \to c} f(x)}$

and $\lim\limits_{x \to c} \sqrt[n]{f(x)}$ are real numbers.

EXAMPLE 2   Using the Limit Properties

You will learn in Example 10 that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ Use this fact, along with the limit properties, to find the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{x + \sin x}{x}$ **(b)** $\lim\limits_{x\to 0} \dfrac{1 - \cos^2 x}{x^2}$ **(c)** $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{\sin x}}{\sqrt[3]{x}} = \sqrt[3]{\dfrac{\sin x}{x}}$

$\lim\limits_{x\to 0} \left(\dfrac{x}{x}\right) + \dfrac{\sin x}{x}$

$\downarrow \qquad \downarrow$

$1 + 1$

$\boxed{2}$

$\lim\limits_{x\to 0} \dfrac{\sin^2 x}{x^2} = \left(\dfrac{\sin x}{x}\right)\left(\dfrac{\sin x}{x}\right)$

$1 \cdot 1$

$\boxed{1}$

$\sqrt[3]{1}$

$\boxed{1}$

Limits of Continuous Functions

Recall from Section 1.2 that a function is continuous at a if $\lim_{x \to a} f(x) = f(a)$. This means that the limit (at a) of a function can be found by "plugging in a" provided the function is continuous at a. (The condition of continuity is essential when employing this strategy. For example, plugging in 0 does not work on any of the limits in Example 2.)

EXAMPLE 3   Finding Limits by Substitution

Find the limits.

(a) $\lim_{x \to 0} \dfrac{e^x - \tan x}{\cos^2 x}$

$$\frac{1-0}{1}$$

$$\boxed{1}$$

(b) $\lim_{n \to 16} \dfrac{\sqrt{n}}{\log_2 n} = \dfrac{\sqrt{16}}{\log_2 16} = \dfrac{4}{4} = \boxed{1}$

One-sided and Two-sided Limits

We can see that the limit of the function in Figure 10.11 is 3 whether x approaches 1 from the left or right. Sometimes the values of a function f can approach different values as x approaches a number c from opposite sides. When this happens, the limit of f as x approaches c from the left is the **left-hand limit** of f at c and the limit of f as x approaches c from the right is the **right-hand limit** of f at c. Here is the notation we use:

left-hand: $\qquad \lim_{x \to c^-} f(x) \qquad$ *The limit of f as x approaches c from the left.*

right-hand: $\qquad \lim_{x \to c^+} f(x) \qquad$ *The limit of f as x approaches c from the right.*

EXAMPLE 4   Finding Left- and Right-Hand Limits

Find $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^{\oplus}} f(x)$ where $f(x) = \begin{cases} -x^2 + 4x - 1 & \text{if } x \le 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$.

left side

right side

$-(2)^2 + 4(2) - 1$

$-4 + 8 - 1$

$\boxed{3}$

$2(2) - 3$

$4 - 3$

$\boxed{1}$

THEOREM One-sided and Two-sided Limits

A function $f(x)$ has a limit as x approaches c if and only if the left-hand and right-hand limits at c exist and are equal. That is,
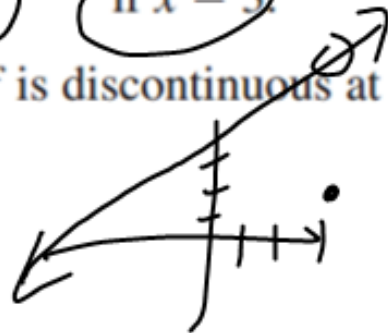
$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L.$$

EXAMPLE 5   Finding a Limit at a Point of Discontinuity

Let

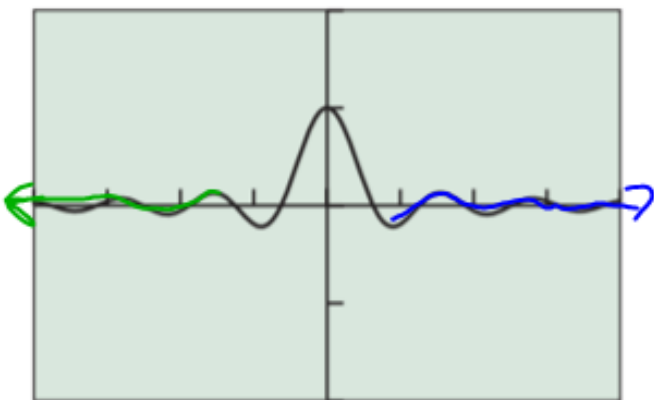$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$$

Find $\lim_{x \to 3} f(x)$ and prove that f is discontinuous at $x = 3$.

factor

$\lim_{x \to 3}$ $\dfrac{(x+3)(x-3)}{(x-3)}$

$3 + 3 \neq 6$

EXAMPLE 7 Investigating Limits as $x \to \pm\infty$

Let $f(x) = (\sin x)/x$. Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
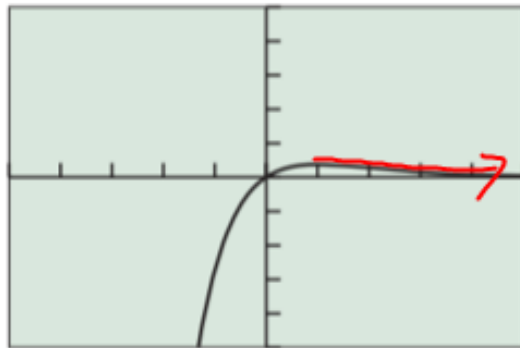
EXAMPLE 8 Using Tables to Investigate Limits as $x \to \pm\infty$

Let $f(x) = xe^{-x}$. Find $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$. $= -\infty$

[−5, 5] by [−5, 5]

X	Y₂	
0	0	
10	4.5E−4	
20	4.1E−8	
30	3E−12	
40	2E−16	
50	1E−20	
60	5E−25	

Y₂ = Xe^(−X)

X	Y₂	
0	0	
−10	−2.2E5	
−20	−9.7E9	
−30	−3E14	
−40	−9E18	
−50	−3E23	
−60	−7E27	

Y₂ = Xe^(−X)

EXAMPLE 9   Investigating Unbounded Limits

Find $\lim\limits_{x \to 2} 1/(x-2)^2.$ $= \infty$

[−4, 6] by [−2, 10]

EXAMPLE 10   Investigating a Limit at $x = 0$

Find $\lim\limits_{x \to 0} (\sin x)/x. = 1$

X	Y₁	
-.03	.99985	
-.02	.99993	
-.01	.99998	
0	ERROR	
.01	.99998	
.02	.99993	
.03	.99985	

Y₁ = sin(X)/X